Error Estimates for a Multidimensional Meshfree Galerkin Method with Diffuse Derivatives and Stabilization

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Abstract

A meshfree method with diffuse derivatives and a penalty stabilization is developed. An error analysis for the approximation of the solution of a general elliptic differential equation, in several dimensions, with Neumann boundary conditions is provided. Theoretical and numerical results show that the approximation error and the convergence rate are better than the diffuse element method.

Key words: Meshfree methods, diffuse derivatives, moving least squares, diffuse element method and error estimates.

Highlights
• Meshfree Methods. • Error estimates for diffuse derivatives. • Diffuse element method modified.

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Estimativos de error para un método de Galerkin libre de mallas en múltiples dimensiones con derivadas difusas y estabilización

Resumen

Se presenta un método libre de mallas con derivadas difusas y estabilización por penalización. Un análisis de error para la aproximación de la solución de una ecuación elíptica general en múltiples dimensiones, con condiciones de frontera tipo Neumann es desarrollado. Resultados numéricos y teóricos muestran que el error de aproximación y la velocidad de convergencia son mejores que en el método de elementos difusos.

Palabras clave: Método libre de mallas, derivadas difusas, mínimos cuadrados en movimiento, método de elementos difusos, estimativos de error.

1 Introduction

Numerical methods based on moving least square (MLS) approximations and Galerkin formulations form a popular class of meshfree schemes. However, the high computational expense in the evaluation of the shape functions and their derivatives are drawbacks to the Galerkin approach. For this purpose Belytschko *et al.* [1] and Breitkopf *et al.* [2] have introduced efficient computational approaches for the evaluation of the MLS shape functions and their derivatives.

An alternative for the computation of derivatives, the diffuse derivative, was used by Nayroles in [3] in the DEM. In the diffuse derivative approximation, only the derivatives of the polynomial basis need to be included in computing the gradients of the local field variables. Belytshko *et al.* [4],[5] argued that diffuse derivatives are not attractive in Galerkin methods because they degrade the accuracy due to their lack of integrability. However, recently, the diffuse derivative has been used in a class of novel meshfree methods (Huerta *et al* [6]) for Stokes problems. Because of their simplicity, diffuse derivatives, unlike the full derivatives, retain the same subspace structure as their defining functions. This special feature allowed Huerta *et al* [6] to circumvent the complicated

incompressibility constraint and define a class of divergence free meshfree approximation functions. Beyond fluid mechanics, we think this new approach could be used to enhance the common mixed method approach.

There are very few papers on meshfree methods with a complete error analysis [7]; among them we note [8] on the RKPM, [9] on the EFG and [10] on the MPCM. The last paper provides a complete mathematical analysis of the MPCM with diffuse derivatives, applied to a Poisson problem with Dirichlet boundary conditions. However it was not until our previous work in [11] that a paper with a complete error analysis of a Galerkin meshfree method with diffuse derivatives (DEM) was done.

In this paper, we introduce an extension of our previous work in [11] where we introduced a Galerkin MLS scheme developed entirely with diffuse derivatives, that we called Stabilized Diffuse Galerkin Method (SDGM). That previous work was completely applied to a 1D problem. We used a novel stabilization procedure and, unlike any of the previous work on diffuse derivative schemes for differential equations, provided a full error analysis of the new approach as well as example computations. The new scheme, when applied to self-adjoint elliptic problems, leaded to fully invertible symmetric positive matrices and has rates of convergence that improve as polynomial degree m is increased. Now in this paper we provided the necesary modificatios to extend these results to a general elliptic differential equation in any dimensions. As we will show in sections 3, 4 and 5, additional consideretions have to be made.

The use of stabilization (or penalty) terms is not uncommon; they have been used in finite element methods [12],[13] and have been proved to be quite effective and, indeed, they are often introduced intuitively. For instance, Beissel and Belytschko [14] introduce a penalty term to stabilize nodal integration in the EFG. We will show that our SDGM is accurate and its rate of convergence increases as the approximating polynomial degree increases and the width of the support domain (R) decreases.

This paper is organized as follows. In section 2 we introduce the model probles. In section 3 we review some aspects of MLS approximations. In section 4 we describe the diffuse derivatives and

prove an approximation theorem. In section 5, we introduce the SDGM and prove our main error estimate. Section 6 gives numerical results and provides a comparison with the theoretical results.

2   Model Problem

For a bounded domain Ω with C^1-boundary $\partial\Omega$ we consider problems of the form:

$$
\begin{cases}
-\displaystyle\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)(x)+c(x)u=f(x) & x\in\Omega, \\
\displaystyle\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i(x)=g(x) & x\in\partial\Omega,
\end{cases}
\tag{1}
$$

where $a_{ij}, c \in L^\infty(\Omega), f \in L^2(\Omega), g \in L^2(\partial\Omega), a_{ij} \in L^\infty(\partial\Omega)$ and ν is a unit normal vector to $\partial\Omega$.

Here $c(x) \geq c_1 > 0$ for all $x \in \Omega$ and $A(x) = (a_{ij}(x))_{n\times n}$ is assumed to be uniformly elliptic in Ω, i.e. there exists $\theta > 0$ such that for all $x \in \Omega$ and all $\eta \in \mathbb{R}^n$,

$$
\sum_{i,j=1}^{n}a_{ij}(x)\eta_i\eta_j \geq \theta|\eta|^2.
$$

After multiplying this differential equation by function v and using Green's formulas we find that:

$$
B(u,v) = F(v)
$$

with

$$
B(u,v) = \int_\Omega\left(\sum_{i,j=1}^{n}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+cuv\right)dx \quad\text{and}
$$

$$
F(v) = \int_\Omega fvdx + \int_{\partial\Omega}gvds.
$$

And this problem has a unique solution by the Lax-Milgram theorem.

We have used Neumann or flux boundary conditions here for simplicity; they are natural and do not need to be imposed explicitly. We expect that Dirichlet conditions could be implemented in this framework using, for instance, the Nitsche approach (See, perhaps, [6]). But, we do not pursue that here in order to focus on this new diffuse derivative approach.

We, further, expect that this approach could be extended to problems with higher order derivatives where the payoff in computational cost from the use of diffuse derivatives instead of full derivatives would be more pronounced.

3 Preliminaries on moving least squares (MLS)

3.1 The moving least squares method

Let $\Lambda = \{x_1, x_2, ..., x_N\}$ be a set of N distinct points inside and in the boundary of $\Omega \subset \mathbb{R}^n$ which is an open and bounded set with Lipschitz boundary $\partial\Omega$ and $u_1, u_2, ...u_N$ be the values of an unknown scalar function $u(x)$ at the points in Λ (i.e. $u_i = u(x_i), 1 \le i \le N$). Also let $R > 0$ (usually called dilation parameter) and consider a positive even weight function $W(x)$ with compact support in $\overline{B_1(0)}$ and $\int_{\mathbb{R}^n} W \, dx \cong 1$. Define $W_R(x) := W(x/R)$ and note W_R has compact support in $\overline{B_R(0)}$.

Let $m << N$ and $\mathbf{p}(z) = \{p_0(z), p_1(z), p_2(z), ..., p_s(z)\}^T$ be a basis of the subspace of polynomials of degree less or equal than m (denoted $\mathcal{P}_m$) in $\mathbb{R}^n$, placed in multi-index ordering. Note that $s+1 = (n+m)!/(n!m!)$. For each $x \in \Omega$ consider

$$P_u(x,y) = \sum_{k=0}^{s} a_k(x) p_k\left(\frac{y-x}{R}\right) = \mathbf{p}^T\left(\frac{y-x}{R}\right)\mathbf{a}(x), \qquad (2)$$

where $\mathbf{a}(x) = \{a_0(x), a_1(x), a_2(x), ..., a_s(x)\}^T$ is chosen such that it

minimizes the functional

$$J(\mathbf{a(x)}) = \sum_{j=1}^{N} W\left(\frac{x_j - x}{R}\right) (P_u(x, x_j) - u(x_j))^2$$

$$= \sum_{j=1}^{N} W_R(x_j - x)(P_u(x, x_j) - u(x_j))^2,$$

for a fixed x.

For each fixed x, $P_u(x, y)$ is a polynomial in y of degree less than or equal to m that represents the best local approximation of $u(x)$ for y in a small neighborhood of width $2R$ of x. Then, since the weight function W usually favors the points closer to x it is natural to define the following approximation of $u(x)$:

$$u_R(x) = \lim_{y \to x} P_u(x, y) = P_u(x, x) = \mathbf{p}^T(0)\mathbf{a}(x).$$

A short calculation ([15]) shows that

$$\mathbf{a}(x) = M^{-1}(x)B(x)U,$$

where

$$M(x) = \sum_{x_i \in \Lambda(x)} W\left(\frac{x_i - x}{R}\right) \mathbf{p}\left(\frac{x_i - x}{R}\right) \mathbf{p}^T\left(\frac{x_i - x}{R}\right), \quad U = [u_1, u_2, ..., u_N]^T$$

and $B(x)$ is a matrix whose ith column is $\mathbf{p}((x_i - x)/R) W((x_i - x)/R)$.

Thus,

$$P_u(x, y) = \sum_{i=1}^{N} \mathbf{p}^T\left(\frac{y - x}{R}\right) M^{-1}(x)\mathbf{p}\left(\frac{x_i - x}{R}\right) W\left(\frac{x_i - x}{R}\right) u_i. \quad (3)$$

Therefore, letting $y \mapsto x$

$$u(x) \approx u_R(x) = P_u(x, x) = \sum_{i=1}^{N} \mathbf{p}^T(0)M^{-1}(x)\mathbf{p}\left(\frac{x_i - x}{R}\right) W\left(\frac{x_i - x}{R}\right) u_i.$$
$$(4)$$

This expression can be written in the standard interpolation form

$$u_R(x) = \sum_{i=1}^{N} \varphi_i(x) u_i, \tag{5}$$

where $\varphi_i(x), 1 \leq i \leq N$, are called the shape functions and are given by

$$\varphi_i(x) = \mathbf{p}^T(0) M^{-1}(x) \mathbf{p}\left(\frac{x_i - x}{R}\right) W\left(\frac{x_i - x}{R}\right), \tag{6}$$

and $u_R(x)$ corresponds to the moving least squares approximation of the function u at the point $x \in \Omega$.

Note also that from the minimization of functional $J(\mathbf{a(x)})$ and, in computing the minimum via partial derivatives, the following orthogonality relationship holds:

$$\sum_{j=1}^{N} W\left(\frac{x_j - x}{R}\right) (P_u(x, x_j) - u(x_j)) q\left(\frac{x_j - x}{R}\right) = 0 \tag{7}$$

where $q = q(z)$ is a polynomial of degree less than or equal to m.

3.2 Some error estimates for MLS approximations

We use the notation $W^{m,p}(\Omega)$ to denote the Sobolev space consisting of functions with m derivatives in $L^p(\Omega)$ ($1 \leq p \leq \infty$), and $H^m(\Omega)$ for the special case where $p = 2$. We use the following notation for the norms and seminorms:

$$\|u\|_{L^p(\Omega)} = \int_{\Omega} |u|^p \, dx, \quad \|u\|_{L^\infty(\Omega)} = \inf\{\sigma \geq 0, |u(x)| \leq \sigma, \text{a.e. } x \in \Omega\},$$

$$\|u\|_{W^{m,p}(\Omega)} = \sum_{l=0}^{m} \int_{\Omega} \left|\frac{d^m u}{dx^m}\right|^p \, dx \text{ and } |u|_{W^{m,p}(\Omega)} = \int_{\Omega} \left|\frac{d^m u}{dx^m}\right|^p \, dx.$$

We will also denote $\|v\| := \|v\|_{L^2(\Omega)}$.

The moving least squares method (MLS) has been used for the numerical solution of ordinary and partial differential equations in many papers, but very few of these include error estimates. The papers [9] and [8] provide such analysis and require the following properties:

(i) For each $x \in \Omega$ there exists at least $s + 1$ distinct points from Λ in $B_{\frac{R}{2}}(x)$, where Lagrange interpolation is possible.

(ii) There exists $c_0 > 0$, independent of R, such that $W_R(z) \geq c_0$ for all $z \in B_{\frac{R}{2}}(0)$.

(iii) There exists $c_{\#}$ such that for all $x \in \Omega, card\{x_j \in B_R(x), 1 \leq j \leq N\} < c_{\#}$.

(iv) For any $x \in \Omega$ there exists a constant c_L such that the Lagrange basis functions associated with the set the points in property (i) are bounded by c_L in $B_{2R}(x)$.

(v) $W_R \in C^1(B_R(0)) \cap W^{1,\infty}(\mathbb{R})$ and there exists $c_1 > 0$ such that $||\nabla W_R(z)||_{L^\infty(\mathbb{R}^n)} \leq c_1/R$.

Note, in particular, that $c_0, c_{\#}, c_L$ and c_1 are independent of R and N. Here (iii) implies that $card(\Lambda(x)) \leq c_{\#}$. Throughout the rest of this paper, C will denote a positive constant that is independent of R (and thus N as well).

Remark 3.1. In [8], Han and Meng stated that if conditions (ii) and (iv) are satisfied, then the family of particle distributions, Λ, is called regular, and it is enough to guarantee that there exists a constant L_0 such that $\max_{x \in \overline{\Omega}} ||M(x)^{-1}||_2 \leq L_0$.

With these assumptions the following theorem was established (see proof in [9], [16], [8], [17], [15]):

Theorem 3.1. If properties (i)-(v) hold and $V \in H^{m+1}(\Omega)$, then there exists a constant C_1 that depends on $c_0, c_{\#}, c_L$ and a constant C_2 that depends on $c_0, c_{\#}, c_L, c_1$ such that

$$||V - V_R|| \leq C_1 R^{m+1}|u|_{H^{m+1}(\Omega)}$$

and

$$||\nabla V - \nabla V_R|| \leq C_2 R^m |u|_{H^{m+1}(\Omega)}.$$

4 The diffuse derivative

Computation of the derivatives of a MLS function involves differentiation of $a(x)$ which, in turn, involves differentiation of the M^{-1} and B matrices.

On the other hand, the derivative of polynomials in $\mathcal{P}_m$ is trivial and can be evaluated a priori. The concept of diffuse derivative, proposed in [3], exploits the fact that, for a MLS function u, $P_u(x, y)$ is a good approximation of $u = u(y)$ near the point x. Thus, in the one-dimensional case:

$$\delta u_R(x) = \lim_{y \to x} \frac{\partial P_u(x,y)}{\partial y} = \lim_{y \to x} \frac{\partial \mathbf{p}^T ((y-x)/R)}{\partial y} \mathbf{a}(x) = \sum_{i=1}^{N} \delta\varphi_i(x) u_i, \tag{8}$$

where

$$\delta\varphi_i(x) = \frac{1}{R} [\mathbf{p}'(0)]^T M^{-1}(x) W_R (x_i - x) \mathbf{p}\left(\frac{x_i - x}{R}\right).$$

With obvious modifications for the multidimensional case, using multi-index notation. Below we indicate why $u'(x) \approx \delta u(x)$. It can be shown (see [18]):

Proposition 4.1. Assume $W(x) \in C_0(\mathbb{R}^n)$ and $v(x) \in C^{m+1}(\overline{\Omega})$ where Ω is a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary, and suppose $\sup(\varphi_i)$ is convex for each i. Then if $m > \frac{n}{p} - 1$,

$$||D^\beta v - \delta^\beta v||_{L^p(\Omega)} \leq C(m) R^{m+1-|\beta|} ||v||_{W^{m+1,p}(\Omega)} \quad \forall\, 0 \leq |\beta| \leq m$$

(where $D^\beta v$ and $\delta^\beta v$ represent the full and diffuse derivatives, of order $|\beta|$ of v in multi-index notation).

Now consider a function $V \in \mathcal{V}_R = span\{\varphi_1, \varphi_2, ..., \varphi_N\}$ defined as

$$V(x) = \sum_{i=1}^{N} \varphi_i(x)\overline{v}_i, \tag{9}$$

where $\overline{v}_i, 1 \leq i \leq N$, are constants and $\varphi_i(x)$ are the MLS shape functions. Note that V can also been written as

$$V(x) = \lim_{y \to x} P_{\overline{V}}(x, y) = \lim_{y \to x} \mathbf{p}^T \left(\frac{y - x}{R}\right) \mathbf{a}(x) = \mathbf{p}^T(0)\mathbf{a}(x) \tag{10}$$

with

$$\mathbf{a}(x) = \sum_{j=1}^{N} M^{-1}(x)\mathbf{p}\left(\frac{x_j - x}{R}\right) W_R(x_j - x)\overline{v}_j.$$

The analysis in [9] provides arguments to show that derivatives of the $\mathbf{a}(x)$ functions are small, which is crucial in our theory. The next lemma shows how the diffuse derivative is controlled by differences with the local MLS functional $P_{\overline{V}}(x, y)$. This lemma, somewhat of an inverse estimate, provides the key idea behind our stabilization.

Lemma 4.1. If properties (i)-(v) hold and $|\alpha| = 1$ then there exists a constant C, independent of R, such that

$$||D^\alpha V - \delta^\alpha V||^2 \leq \frac{C}{R^2} \int_\Omega \sum_{x_k \in \Lambda(x)} |P_{\overline{V}}(x, x_k) - \overline{v}_k|^2 dx.$$

where $\Lambda(x) = \{x_j \in \Lambda : x \in \overline{B_R(x_j)} \cap \overline{\Omega}\}$.

Proof. Our proof follows very closely the proof of lemma 2.2 in [9] and Lemma 3 in [11].

Let $\alpha = (0, \cdots, 0, \alpha_j, 0 \cdots, 0) \in \mathbb{R}^n$, $\alpha_j = 1$. So

$$D^\alpha v = \frac{\partial v}{\partial x_j} \quad \text{and} \quad \delta^\alpha v = \frac{\delta v}{\delta x_j} = \delta_{x_j} v$$

Let also, $e_j = $ jth unit vector in $\mathbb{R}^n$.

A short calculation shows that

$$D^\alpha V(x) - \delta^\alpha V(x)$$

$$= \lim_{y \to x} \left[\lim_{h \to 0} \frac{1}{h} \sum_{k=0}^{s} \left(a_k(x + he_j)p_k \left(\frac{y-x}{R} \right) - a_k(x)p_k \left(\frac{y + he_j - x}{R} \right) \right) \right].$$

So,

$$D^\alpha V(x) - \delta^\alpha V(x) = \lim_{y \to x} \left[\lim_{h \to 0} \frac{1}{h} \left(P_{\overline{V}}(x + he_j, y + he_j) - P_{\overline{V}}(x, y + he_j) \right) \right].$$

Define $Q(x, y) := (P_{\overline{V}}(x + he_j, y) - P_{\overline{V}}(x, y))$.

$$D^\alpha v(x) - \delta^\alpha v(x) = \lim_{y \to x} \left[\lim_{h \to 0} \frac{Q(x, y + he_j)}{h} \right]. \tag{11}$$

Notice that for each $x, Q(x, y)$ is a polynomial in y of degree less or equal than m. Also, if $z_1, z_2, ..., z_{s+1}$ are the points in $B_{R/2}(x)$ given by property (i), then using property (ii),

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{1}{c_0} \sum_{l=1}^{N} W \left(\frac{x_l - x}{R} \right) |P_{\overline{V}}(x + he_j, x_l) - P_{\overline{V}}(x, x_l)|^2$$

$$= \frac{1}{c_0} \sum_{l=1}^{N} W \left(\frac{x_l - x}{R} \right) Q(x, x_l)[P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l]$$

$$+ \frac{1}{c_0} \sum_{l=1}^{N} W \left(\frac{x_l - x}{R} \right) Q(x, x_l)[\overline{v}_l - P_{\overline{V}}(x, x_l)].$$

By the orthogonality property (7) the second term on the right hand side above is zero. Thus

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{1}{c_0} \sum_{l=1}^{N} W \left(\frac{x_l - x}{R} \right) Q(x, x_l)[P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l].$$

By property (iv) and the Meanvalue theorem, we can guarantee that, for h small enough, $\exists \xi_l$ such that

$$W\left(\frac{x_l - x}{R}\right) - W\left(\frac{x_l - x - he_j}{R}\right) = \frac{\partial W}{\partial x_j}(\xi_l)h$$

so,

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{1}{c_0} \sum_{l=1}^{N} W\left(\frac{x_l - x - he_j}{R}\right) Q(x, x_l)[P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l]$$

$$+ \frac{h}{c_0} \sum_{l=1}^{N} \frac{\partial W}{\partial x_j}(\xi_l)Q(x, x_l)[P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l],$$

and using again the orthogonality condition (in $x + he_j$) and property (v), we have

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{h}{c_0} \sum_{l=1}^{N} \frac{\partial W}{\partial x_j}(\xi_l)Q(x, x_l)[P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l]$$

$$\leq \frac{c_1 h}{c_0 R} \sum_{x_l \in \Lambda_{1+h}(x)} |Q(x, x_l)||P_{\overline{V}}(x + he_j, x_l) - \overline{v}_l| \tag{12}$$

where $\Lambda_{1+h}(x) = \{x_k \in \Lambda : |x - x_k| \leq (1+h)R\}$. On the other hand, we can use the Lagrange's polynomial basis functions $l_0, l_1, \ldots, l_s$, each of degree m at the $s + 1$ distinct points in $B_{R/2}(x)$ to write

$$Q(x, w) = \sum_{z_i \in B_{R/2}(x)} Q(x, z_i)l_i(w). \tag{13}$$

So, using the Cauchy-Schwarz inequality,

$$\sum_{x_k \in \Lambda_{1+h}(x)} |Q(x, x_k)|^2 \leq \sum_{x_k \in \Lambda_{1+h}(x)} \left(\sum_{z_i \in B_{R/2}(x)} |Q(x, z_i)||l_i(x_k)|\right)^2$$

$$\leq \left(\sum_{z_i \in B_{R/2}(x)} |Q(x, z_i)|^2\right) \left(\sum_{x_k \in \Lambda_{1+h}(x)} \sum_{z_i \in B_{R/2}(x)} |l_i(x_k)|^2\right)$$

$$\leq C(c_L, c_\#, m) \sum_{z_i \in B_{R/2}(x)} |Q(x, z_i)|^2.$$

Using this result and Cauchy-Schwarz inequality in equation (12) we have

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{Ch}{R} \left(\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \right)^{1/2} \cdot$$

$$\left(\sum_{x_k \in \Lambda_{1+h}(x)} |P_{\overline{V}}(x + he_j, x_k) - \overline{v}_k|^2 \right)^{1/2}$$

and thus,

$$\sum_{z_k \in B_{R/2}(x)} |Q(x, z_k)|^2 \leq \frac{Ch^2}{R^2} \sum_{x_k \in \Lambda_{1+h}(x)} |P_{\overline{V}}(x + he_j, x_k) - \overline{v}_k|^2. \quad (14)$$

Using the equation (14), choosing $z = y + he_j$ and property (iv) in (13) we obtain, for all $y \in B_R(x) \cap \Omega$,

$$\left| \frac{Q(x, y + he_j)}{h} \right|^2 \leq C \sum_{z_k \in B_{R/2}(x)} \left| \frac{Q(x, z_k)}{h} \right|^2$$

$$\leq \frac{C}{R^2} \sum_{x_k \in \Lambda_{1+h}(x)} |P_{\overline{V}}(x + he_j, x_k) - \overline{v}_k|^2.$$

Finally, using this result in the equation (11) yields:

$$\|D^\alpha v(x) - \delta^\alpha v(x)\|^2 \leq \frac{C}{R^2} \int_\Omega \sum_{x_k \in \Lambda(x)} |P_{\overline{V}}(x, x_k) - \overline{v}_k|^2 dx. \qquad \square$$

Remark 4.1. In the one dimensional case we presented in [15] and [11], we had that if $V = u_R$ (i.e. $\overline{v}_i = u_i = u(x_i)$) then $P_u(x, \cdot)$ is an interpolating polynomial of the function u at least on $m + 1$ distinct

points in $\Lambda(x)$; however, in the multidimensional case we are not aware of a similar result or whether this is true or not. Therefore, let us introduce for any $x \in \Omega$ the polynomial $I_u(x, \cdot)$ in $\mathcal{P}_m$ that interpolates u at the points that satisfy the property (i). We can prove the following theorem (see [9] or [15]):

Theorem 4.1. Let $x \in \Omega$; if properties (i)-(iv) hold, then there exists a constant C depending only on $c_0, c_\#, c_L$ such that for all $y \in B_R(x) \cap \Omega$

$$|u(y) - P_u(x,y)| \leq C\|u - I_u(x, \cdot)\|_{L^\infty(B_R(x) \cap \Omega)}. \tag{15}$$

Therefore, in particular taking $y = x$ we have

$$|u(y) - u_R(x)| \leq C\|u - I_u(x, \cdot)\|_{L^\infty(B_R(x) \cap \Omega)}. \tag{16}$$

5   A Galerkin Approximation Scheme

The stabilization of our Galerkin scheme will involve

$$\mathcal{P}(U,V) = \int_\Omega \left[\sum_{x_l \in \Lambda(x)} (P_{\overline{U}}(x, x_l) - \overline{u}_l)(P_{\overline{V}}(x, x_l) - \overline{v}_l) \right] dx.$$

where U and V are MLS functions in $\mathcal{V}_R$ as defined in (9). So, by lemma 4.1 we have

$$\|D^\alpha V - \delta^\alpha V\|^2 \leq \frac{C}{R^2} \mathcal{P}(V, V). \tag{17}$$

Let us also notice that from theorem 4.1 we have

$$|P_u(x, x_l) - u_l| = |P_u(x, x_l) - u(x_l)| \\ \leq C\|u - I_u(x, \cdot)\|_{L^\infty(\Omega)}.$$

Thus, by standard arguments from polynomial interpolation

$$|P_u(x, x_l) - u_l| \leq CR^{m+1}\|u^{m+1}\|_{L^\infty(\Omega)}.$$

Hence,

$$\mathcal{P}(u_R, u_R) = \int_\Omega \left[\sum_{x_l \in \Lambda(x)} (P_u(x, x_l) - u(x_l))^2 \right] dx$$

$$\leq C \int_\Omega \sum_{x_l \in \Lambda(x)} R^{2(m+1)} \|u^{m+1}\|_{L^\infty(\Omega)}^2 dx$$

$$\leq C R^{2(m+1)} \|u^{m+1}\|_{L^\infty(\Omega)}^2. \tag{18}$$

For MLS functions u and w, the bilinear form for our Galerkin scheme is

$$B(u, w) = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x) \frac{\delta u}{\delta x_j} \frac{\delta w}{\delta x_i} + c(x) u v \right) dx := (A \delta u, \delta w) + (cu, w) \tag{19}$$

where

$$\delta u = \left[\frac{\delta u}{\delta x_1}, \frac{\delta u}{\delta x_2}, \cdots, \frac{\delta u}{\delta x_n} \right]^T$$

and A is a matrix whose ij entry is a_{ij}

So, our stabilized diffuse Galerkin method (SDGM) is as follows:

SDGM: $\begin{cases} \text{Find } U \in \mathcal{V}_R \ \text{ so that} \\ B(U, \beta) + R^{-2\gamma}\mathcal{P}(U, \beta) = (f, \beta) \quad \forall \, \beta \in \mathcal{V}_R. \end{cases}$

Here, we require $\gamma > 0$ and guidelines for it will be provided in the next theorem.

$\square$

Remark 5.1. Note that **SDGM** is equivalent to the following Ritz-Galerkin formulation: Find $U \in \mathcal{V}_R$ so that,

$$J(U) = \min_{V \in \mathcal{V}_R} J(V) = \min_{V \in \mathcal{V}_R} [B(V, V) - 2(f, V) + R^{-2\gamma}\mathcal{P}(V, V)],$$

from which we can identify $R^{-2\gamma}\mathcal{P}(V,V)$, as a penalty or stabilization
term.

We can now state and prove our main theorem:

Theorem 5.1. Let $u \in C^{m+1}(\overline{\Omega})$ be the exact solution of the equation
(1), u_R be its MLS approximation and U be the solution given by the
numerical scheme **SDGM**, then if $\gamma = m/2 + 1$, there exists a constant
C independent of R such that,

$$\|\delta u - DU\| + \|u - U\| \leq CR^{m/2}, \tag{20}$$

where

$$Du = \left[\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\right]^T$$

Proof. The proof of this theorem is similar to the proof of theorem 1 in
[11].

First, for $\beta \in \mathcal{V}_R$

$$\begin{aligned}
B(u_R, \beta) &= (ADu, \delta\beta) + (cu, \beta) + (A(\delta u - Du), \delta\beta)\\
&= (f, \beta) + (ADu, \delta\beta - D\beta) + (A(\delta u_R - Du), \delta\beta), \tag{21}
\end{aligned}$$

Now, let $e = u_R - U$, then by the equations **SDGM** and (21) we
have,

$$\begin{aligned}
B(e, \beta) - R^{-2\gamma}\mathcal{P}(U, \beta) &= (ADu, \delta\beta - D\beta) + (A(\delta u_R - Du), \delta\beta)\\
&\quad + (c(u_R - u), \beta). \tag{22}
\end{aligned}$$

Now, choosing $\beta = e$ we find

$$\begin{aligned}
B(e, e) + R^{-2\gamma}\mathcal{P}(U, U) &= (ADu, \delta e - De) + (A(\delta u_R - Du), \delta e)\\
&\quad + (c(u_R - u), e) + R^{-2\gamma}\mathcal{P}(U, u_R).
\end{aligned}$$

We can use the ellipticity condition to show

$$B(e, e) \geq \theta\|\delta e\|^2 + c_1\|e\|^2.$$

Therefore

$$\theta\|\delta e\|^2 + c_1\|e\|^2 + R^{-2\gamma}\mathcal{P}(U,U)$$
$$\leq (ADu, \delta u_R - Du_R) + (A(\delta u_R - Du), \delta e)$$
$$+ (c(u_R - u), e) - (ADu, \delta U - DU) + R^{-2\gamma}\mathcal{P}(U, u_R). \tag{23}$$

We can now use the Cauchy-Schwartz and arithmetic-geometric mean inequalities together with theorem 3.1 and proposition 4.1 to find

$$\theta\|\delta e\| + c_1\|e\| + R^{-2\gamma}\mathcal{P}(U,U)$$
$$\leq \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \left(\int_\Omega |Du|(|\delta u_R - Du| + |Du - Du_R|)dx \right.$$
$$\left. + \int_\Omega |\delta u_R - Du||\delta e|dx + \int_\Omega |Du||\delta U - DU|dx \right)$$
$$+ \|c\|_{L^\infty(\Omega)} \int_\Omega |u_R - u||e|dx + R^{-2\gamma}(\mathcal{P}(U,U))^{1/2}(\mathcal{P}(u_R, u_R))^{1/2}$$
$$\leq C\left(\|Du\|\|\delta u_R - Du\| + \|Du\|\|Du - Du_R\| + \|\delta u_R - Du\|\|\delta e\|\right.$$
$$+\|Du\|\|\delta U - DU\|\big) + C\|u_R - u\|\|e\|$$
$$+ R^{-2\gamma}(\mathcal{P}(U,U))^{1/2}(\mathcal{P}(u_R, u_R))^{1/2}$$
$$\leq CR^m + C\|\delta e\|R^m + C\|\delta U - DU\| + C\|e\|R^{m+1}$$
$$+ R^{-2\gamma}(\mathcal{P}(U,U))^{1/2}(\mathcal{P}(u_R, u_R))^{1/2}$$
$$\leq CR^m + \frac{1}{2}\left(\frac{C}{\epsilon}R^{2m} + \epsilon\|\delta e\|^2\right) + \frac{k}{2}\left(\frac{C}{\epsilon}R^{2(m+1)} + \epsilon\|e\|^2\right)$$
$$+ \frac{1}{2}\left(\frac{C}{\epsilon}R^{2(\gamma-1)} + \frac{\epsilon}{C}R^{-2(\gamma-1)}\|\delta U - U'\|^2\right) + \frac{\epsilon}{2}R^{-2\gamma}\mathcal{P}(U,U)$$
$$+ \frac{1}{2\epsilon}R^{-2\gamma}\mathcal{P}(u_R, u_R).$$

Now, by (18) and (17) we have

$$\left(\theta - \frac{\epsilon}{2}\right)\|\delta e\|^2 + c_1\left(1 - \frac{\epsilon}{2}\right)\|e\|^2 + (1 - \epsilon)R^{-2\gamma}\mathcal{P}(\mathcal{M}(U), \mathcal{M}(U))$$

$$\leq C \left[R^m + \frac{1}{2\epsilon} R^{2m} + \frac{k}{2\epsilon} R^{2(m+1)} + \frac{1}{2\epsilon} R^{2(\gamma-1)} + \frac{1}{2\epsilon} R^{2(m-\gamma+1)} \right].$$

Choosing $\epsilon = 1/2$, we conclude

$$\|\delta e\|^2 + \|e\|^2 + R^{-2\gamma} \mathcal{P}(\mathcal{M}(U), \mathcal{M}(U))$$
$$\leq C \left(R^m + R^{2m} + R^{2(m+1)} \right) + C \left(R^{2(\gamma-1)} + R^{2(m-\gamma+1)} \right). \quad (24)$$

Then, taking $\gamma = m/2 + 1$ proves

$$\|\delta e\|^2 + \|e\|^2 + R^{-(m+2)} \mathcal{P}(U, U) \leq C R^m, \quad (25)$$

and since $\mathcal{P}(U, U) \geq 0$ we find that

$$(\|\delta U - \delta u_R\|^2 + \|U - u_R\|^2)^{1/2} \leq C R^m.$$

Combining this with proposition (4.1) we finish the proof. □

6   Numerical results

In this section we present some numerical results on the convergence orders of the **SDGM**. The numerical results confirm the theoretical predictions.

Solutions are reported for the following numerical methods:

1. Diffuse element method (DEM).

2. Element free Galerkin method (EFG).

3. The stabilized diffuse Galerkin method (SDGM).

In all cases a background mesh of subintervals on cells was used for numerical integration. Within each integration cell, there was a set of Gauss-Legendre quadrature points. We kept the number of cells large enough so that numerical integration did not affect the convergence rates. The weight function was chosen to be the cubic spline:

$$W(x) = 2 \begin{cases} 4(|x| - 1)x^2 + 2/3 & |x| \leq 0.5 \\ 4(|x| - 1)^3/3 & 0.5 \leq |x| \leq 1 \\ 0 & 1 \leq |x|. \end{cases} \quad (26)$$

We applied the numerical methods to the following boundary value problem:

$$\begin{cases} -\Delta u + \pi^2 u = 3\pi^2 cos(\pi x)cos(\pi y) & (x,y) \in \Omega, \\ \frac{\partial u}{\partial n} = 0 & (x,y) \in \partial\Omega, \end{cases} \tag{27}$$

where $\Omega = [0,1] \times [0,1]$ and n is the unit normal vector to the boundary of Ω. The exact solution of the equation (27) is $u(x,y) = cos(\pi x)cos(\pi x)$.

For this example, we divide Ω into $11 \times 11, 21 \times 21$ and 31×31 uniformly distributed points, and the dilation parameter (R/h) is kept constant for each m, so that the hypotheses from section 3 are satisfied.
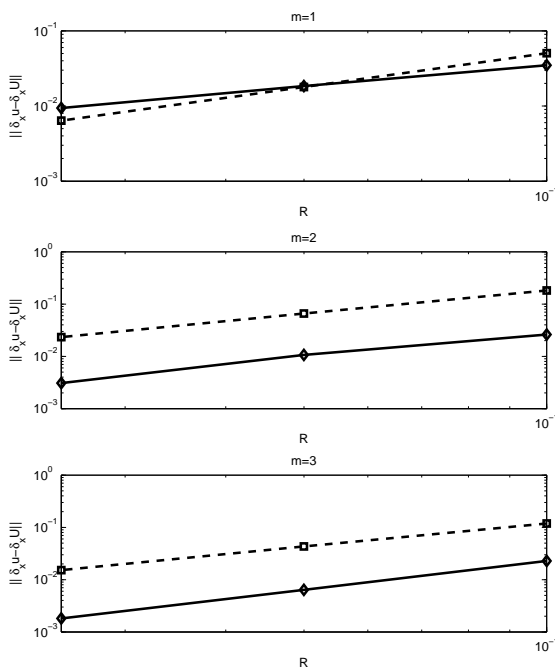

Figure 1: Partial derivative with respect to x, approximation error in L^2 norm. The continuous line with diamonds is for the SDGM and the discontinuous line for the DEM.

Figures 1 and 2 show numerical results comparing EFG, DEM and SDGM for different dimensions (m) of the polynomial basis used in the MLS approximation. We report errors for both the numerical approximation of the solution of the equation (27) and the approximation of its first derivative with respect to x using diffuse derivatives, in the L^2 norm. It is important to notice that similar results can be obtained for the full and diffuse derivatives with respect to y, and therefore for the gradient and diffuse gradient of u.
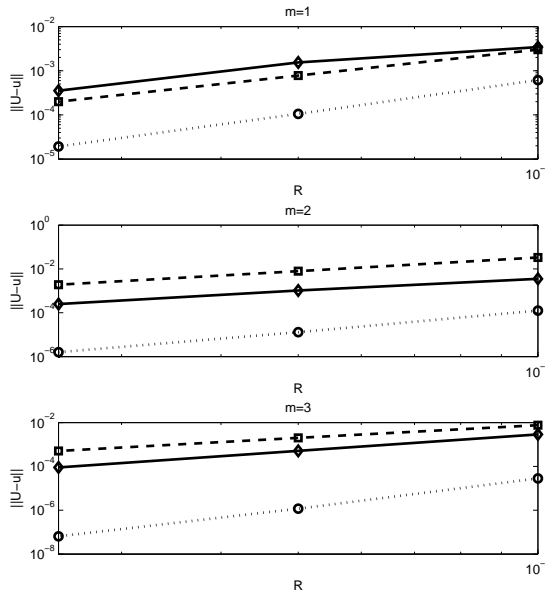

Figure 2: Numerical solution error in L^2 norm. The continuous line with diamonds is for the SDGM, the dashed line with squares for the DEM and the pointed line with circles for the EFG.

The convergence rates are summarized in tables 1 and 2. It can be seen that the numerical results suggest that for the SDGM

$$\|u - U\| \leq CR^{(m+2)/2} \quad \text{and} \quad \|\delta_x u - \delta_x U\| \leq CR^{(m+1)/2}, \qquad (28)$$

while we only get about $\|u - U\| \leq CR^{1.5}$ and $|\delta_x u - \delta_U| \leq CR^{0.5}$ for

the DEM.

Table 1: Convergence Rates for Solution Approximation in L^2

Method	$m = 1$	$m = 2$	$m = 3$
DEM	1.46	1.56	1.49
SDGM	1.63	1.92	2.49
EFG	2.49	3.15	4.10

Table 2: Convergence Rates for Diffuse Derivative Approximation (with respect to x) in L^2

Method	$m = 1$	$m = 2$	$m = 3$
DEM	1.09	1.08	1.03
SDGM	0.95	1.54	1.88

The numerical convergence rates are slightly better than our theoretical results on $||\delta_x u - \delta_x U||$ and $||u - U||$, as is often true. We also observe that, in general, the errors obtained using SDGM are smaller than those using the standard DEM. The convergence rates for the DEM in the L^2 norm are about 1.5 independent of the value of m and 1 for its derivative (similar observations were made in [19] and [11], although a small convergence rate for the DEM was obtained in the 1D case). Our SDGM performs better as m increases. Also, as expected the standard EFG gives the best convergence rates, but unfortunately it is very expensive from the computational point of view, which makes diffuse derivatives more attractive. It is important to notice that all these results agree with those found for the 1D case in [11].

Remark 6.1. A proof of the enhanced L^2 convergence observed in the equation (28) remains an open question. The fact that our penalty term is not exactly zero at the true solution and integration by parts with diffuse derivatives is not possible, seem to make the standard duality argument impossible.

7   Conclusions

A modification to the traditional DEM, the stabilized diffuse Galerkin method (SDGM) has been proposed, in which a stabilization term is introduced to improve the overall accuracy and stability. The new scheme, like DEM, does not require the evaluation of full derivatives. This method is shown to give better results than DEM and converges to the true solution as the dilation parameter (R) goes to zero, or the order of the polynomial basis is increased, as demonstrated numerically and proved theoretically. The procedure described in this paper can be applied to more general multidimensional problems (see [15]).

We see **SDGM** as enhancing the viability of the diffuse derivative approach. Again, as suggested in Huerta *et al* [6], we think the versatility of the diffuse derivative could be helpful in fluid flows or mixed method computations.

Acknowledgements

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