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Generalized Extended Matrix Variate Beta and Gamma Functions and Their Applications

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Abstract

In this article, we define and study generalized forms of extended matrix variate gamma and beta functions. By using a number of results from matrix algebra, special functions of matrix arguments and zonal polynomials we derive a number of properties of these newly defined functions. We also give some applications of these functions to statistical distribution theory.

Key words: Beta function; extended beta function; extended matrix variate beta distribution; extended gamma function; gamma function; matrix argument; zonal polynomial.

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Funciones Beta y Gama generalizadas extendidas y sus aplicaciones

Resumen
En este artículo definimos y estudiamos formas generalizadas de las funciones gama y beta matriz variadas extendidas. Utilizando varios resultados del álgebra matricial, funciones especiales de argumento matricial y polinomios zonales, derivamos algunas de las propiedades de estas funciones. También mostramos algunas aplicaciones de estas funciones a la teoría de distribuciones.

Palabras clave: Función Beta; función Beta extendida; distribución Beta matriz variada extendida; función Gama; función Gama extendida; argumento matricial; polinomio zonal.

1 Introduction

The gamma function was first introduced by Leonard Euler in 1729, as the limit of a discrete expression and later as an absolutely convergent improper integral, namely,

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t)\, \mathrm{d}t, \quad \mathrm{Re}(a) > 0. \tag{1}$$

The gamma function has many beautiful properties and has been used in almost all the branches of science and engineering.

One year later, Euler introduced the beta function defined for a pair of complex numbers a and b with positive real parts, through the integral

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, \mathrm{d}t, \quad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0. \tag{2}$$

The beta function has many properties, including symmetry, $B(a,b) = B(b,a)$, and its relationship to the gamma function,

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

In statistical distribution theory, gamma and beta functions have been used extensively. Using integrands of gamma and beta functions, the gamma and beta density functions are usually defined.

Recently, the domains of gamma and beta functions have been extended to the whole complex plane by introducing in the integrands of (1) and (2), the factors $\exp(-\sigma/t)$ and $\exp[-\sigma/t(1-t)]$, respectively, where $\mathrm{Re}(\sigma) > 0$. The functions so defined have been named extended gamma and extended beta functions.

In 1994, Chaudhry and Zubair [7] defined the extended gamma function, $\Gamma(a;\sigma)$, as

$$\Gamma(a;\sigma) = \int_0^\infty t^{a-1} \exp\left(-t - \frac{\sigma}{t}\right) \mathrm{d}t, \tag{3}$$

where $\mathrm{Re}(\sigma) > 0$ and a is any complex number. For $\mathrm{Re}(a) > 0$ and $\sigma = 0$, it is clear that the above extension of the gamma function reduces to the classical gamma function, $\Gamma(a, 0) = \Gamma(a)$. The extended gamma function is a special case of Krätzel function defined in 1975 by Krätzel [16]. The generalized gamma function (extended) has been proved very useful in various problems in engineering and physics, see for example, Chaudhry and Zubair [8].

In 1997, Chaudhry et al. [6] defined the extended beta function

$$B(a, b;\sigma) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{4}$$

where $\mathrm{Re}(\sigma) > 0$ and parameters a and b are arbitrary complex numbers. When $\sigma = 0$, it is clear that for $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$, the extended beta function reduces to the classical beta function $B(a, b)$.

Recently, Özergin, Özarslan and Altın [24] have further generalized the extended gamma and extended beta functions as

$$\Gamma^{(\alpha,\beta)}(a;\sigma) = \int_0^\infty t^{a-1}\Phi\left(\alpha;\beta;-t - \frac{\sigma}{t}\right) \mathrm{d}t, \tag{5}$$

$$B^{(\alpha,\beta)}(a, b;\sigma) = \int_0^1 t^{a-1}(1-t)^{b-1}\Phi\left(\alpha;\beta;-\frac{\sigma}{t(1-t)}\right) \mathrm{d}t, \tag{6}$$

where $\Phi(\alpha;\beta;\cdot)$ is the type 1 confluent hypergeometric function. The gamma function, the extended gamma function, the beta function, the extended beta function, the gamma distribution, the beta distribution and the extended beta distribution have been generalized to the matrix case in various ways. These generalizations and some of their properties can be found in Olkin [23], Gupta and Nagar [10], Muirhead [18], Nagar, Gupta, and Sánchez [19], Nagar, Roldán-Correa and Gupta [20], Nagar and Roldán-Correa [21], and Nagar, Morán-Vásquez and Gupta [22]. For some recent advances the reader is refereed to Hassairi and Regaig [12], Farah and Hassairi [4], Gupta and Nagar [11], and Zine [25]. However, generalizations of the extended gamma and extended beta functions defined by (5) and (6), respectively, to the matrix case have not been defined and studied. It is, therefore, of great interest to define generalizations of the extended gamma and beta functions to the matrix case, study their properties, obtain different integral representations, and establish the connection of these generalizations with other known special functions of matrix argument.

This paper is divided into seven sections. Section 2 deals with some well known definitions and results on matrix algebra, zonal polynomials and special functions of matrix argument. In Section 3, the extended matrix variate gamma function has been defined and its properties have been studied. Definition and different integral representations of the extended matrix variate beta function are given in Section 4. Some integrals involving zonal polynomials and generalized extended matrix variate beta function are evaluated in Section 5. In Section 6, the distribution of the sum of dependent generalized inverted Wishart matrices has been derived in terms of generalized extended matrix variate beta function. We introduce the generalized extended matrix variate beta distribution in Section 7.

2 Some known definitions and results

In this section we give several known definitions and results. We first state the following notations and results that will be utilized in this and subsequent sections. Let $A = (a_{ij})$ be an $m \times m$ matrix of real or complex numbers. Then, A' denotes the transpose of A; $\operatorname{tr}(A) = a_{11} + \cdots + a_{mm}$; $\operatorname{etr}(A) = \exp(\operatorname{tr}(A))$; $\det(A) =$ determinant of A; $\|A\| =$ spectral norm of A; $A = A' > 0$ means that A is symmetric positive definite, $0 < A < I_m$ means that both A and $I_m - A$ are symmetric positive definite, and $A^{1/2}$ denotes the unique positive definite square root of $A > 0$.

Several generalizations of the Euler's gamma function are available in the scientific literature. The multivariate gamma function, which is frequently used in multivariate statistical analysis, is defined by

$$\Gamma_m(a) = \int_{X>0} \operatorname{etr}(-X) \det(X)^{a-(m+1)/2} \, \mathrm{d}X, \tag{7}$$

where the integration is carried out over $m \times m$ symmetric positive definite matrices. By evaluating the above integral it is easy to see that

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right), \quad \operatorname{Re}(a) > \frac{m-1}{2}. \tag{8}$$

The multivariate generalization of the beta function is given by

$$\begin{aligned} B_m(a,b) &= \int_0^{I_m} \det(X)^{a-(m+1)/2} \det(I_m - X)^{b-(m+1)/2} \, \mathrm{d}X \\ &= \int_{Y>0} \det(Y)^{a-(m+1)/2} \det(I_m + Y)^{-(a+b)} \, \mathrm{d}Y \\ &= \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} = B_m(b,a). \end{aligned} \tag{9}$$

The generalized hypergeometric function of one matrix argument as defined by Constantine [9] and James [15] is

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa(X)}{k!}, \tag{10}$$

where $C_\kappa(X)$ is the zonal polynomial of $m \times m$ complex symmetric matrix X corresponding to the ordered partition $\kappa = (k_1,\dots,k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, $k_1 + \cdots + k_m = k$ and $\sum_{\kappa \vdash k}$ denotes summation over all partitions κ. The generalized hypergeometric coefficient $(a)_\kappa$ used above is defined by

$$(a)_\kappa = \prod_{i=1}^{m} \left(a - \frac{i-1}{2}\right)_{k_i} \tag{11}$$

where $(a)_r = a(a+1)\cdots(a+r-1), r = 1,2,\dots$ with $(a)_0 = 1$. The parameters a_i, $i = 1,\dots,p$, b_j, $j = 1,\dots,q$ are arbitrary complex numbers. No denominator parameter b_j is allowed to be zero or an integer or half-integer $\leq (m-1)/2$. If any numerator parameter a_i is a negative integer,

$a_1 = -r$, then the function is a polynomial of degree mr. The series converges for all X if $p \leq q$, it converges for $||X|| < 1$ if $p = q+1$, and, unless it terminates, it diverges for all $X \neq 0$ if $p > q$.

If X is an $m \times m$ symmetric matrix, and R is an $m \times m$ symmetric positive definite matrix, then the eigenvalues of RX are same as those of $R^{1/2}XR^{1/2}$, where $R^{1/2}$ is the unique symmetric positive definite square root of R. In this case $C_\kappa(RX) = C_\kappa(R^{1/2}XR^{1/2})$ and

$$\begin{aligned} &{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; RX) \\ &\quad = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; R^{1/2}XR^{1/2}). \end{aligned} \tag{12}$$

Two special cases of (10) are the confluent hypergeometric function and the Gauss hypergeometric function denoted by Φ and F, respectively. They are given by

$$\Phi(a; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!}$$

and

$$F(a, b; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!}, \quad \|X\| < 1. \tag{13}$$

The integral representations of the confluent hypergeometric function Φ and the Gauss hypergeometric function F are given by

$$\begin{aligned} \Phi(a; c; X) = \frac{1}{B_m(a, c-a)} &\int_0^{I_m} \operatorname{etr}(SX) \\ &\times \det(S)^{a-(m+1)/2} \det(I_m - S)^{c-a-(m+1)/2} \, \mathrm{d}S \end{aligned} \tag{14}$$

and for $X < I_m$,

$$\begin{aligned} F(a, b; c; X) = \frac{1}{B_m(a, c-a)} &\int_0^{I_m} \det(I_m - XS)^{-b} \\ &\times \det(S)^{a-(m+1)/2} \det(I_m - S)^{c-a-(m+1)/2} \, \mathrm{d}S, \end{aligned} \tag{15}$$

where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$.

For properties and further results on these functions the reader is referred to Herz [13], Constantine [9], James [15], and Gupta and Nagar [10].

The confluent hypergeometric function Φ satisfies the Kummer's relation

$$\Phi(a;c;X) = \operatorname{etr}(X)\Phi(c-a;c;-X). \tag{16}$$

From (15), it is easy to see that

$$F(a,b;c;I_m) = \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)}.$$

Lemma 2.1. *Let Z be an $m\times m$ complex symmetric matrix with $\operatorname{Re}(Z)>0$ and let Y be an $m\times m$ complex symmetric matrix. Then, for $\operatorname{Re}(t)>(m-1)/2$, we have*

$$\begin{aligned}\int_{S>0} &\operatorname{etr}(-ZS)\det(S)^{t-(m+1)/2}C_\kappa(YS)\,\mathrm{d}S\\ &= \Gamma_m(t,\kappa)\det(Z)^{-t}C_\kappa(Z^{-1}Y).\end{aligned} \tag{17}$$

Lemma 2.2. *Let Z be an $m\times m$ complex symmetric matrix with $\operatorname{Re}(Z)>0$ and let Y be an $m\times m$ complex symmetric matrix. Then, for $\operatorname{Re}(t)>k_1+(m-1)/2$, we have*

$$\begin{aligned}\int_{S>0} &\operatorname{etr}(-ZS)\det(S)^{t-(m+1)/2}C_\kappa(S^{-1}Y)\,\mathrm{d}S\\ &= \Gamma_m(t,-\kappa)\det(Z)^{-t}C_\kappa(ZY)\\ &= \frac{(-1)^k\Gamma_m(t)}{(-t+(m+1)/2)_\kappa}\det(Z)^{-t}C_\kappa(ZY).\end{aligned} \tag{18}$$

Lemma 2.3. *Let Y be an $m\times m$ complex symmetric matrix, then for $\operatorname{Re}(a)>(m-1)/2$ and $\operatorname{Re}(b)>(m-1)/2$, we have*

$$\begin{aligned}\int_0^{I_m} &\det(S)^{a-(m+1)/2}\det(I_m-S)^{b-(m+1)/2}C_\kappa(YS)\,\mathrm{d}S\\ &= \frac{\Gamma_m(a,\kappa)\Gamma_m(b)}{\Gamma_m(a+b,\kappa)}C_\kappa(Y).\end{aligned} \tag{19}$$

Lemma 2.4. *Let Y be an $m\times m$ complex symmetric matrix. Then*

$$\int_0^{I_m} \det(R)^{a-(m+1)/2}\det(I_m-R)^{b-(m+1)/2}C_\kappa(R^{-1}Y)\,\mathrm{d}R$$

$$= \frac{\Gamma_m(a,-\kappa)\Gamma_m(b)}{\Gamma_m(a+b,-\kappa)} C_\kappa(Y)$$
$$= \frac{\Gamma_m(a)\Gamma_m(b)(-a-b+(m+1)/2)_\kappa}{(-a+(m+1)/2)_\kappa \Gamma_m(a+b)} C_\kappa(Y), \tag{20}$$

where $\operatorname{Re}(a) > k_1 + (m-1)/2$ *and* $\operatorname{Re}(b) > (m-1)/2$.

Results given in Lemma 2.1 and Lemma 2.3 were given by Constantine [9] while Lemma 2.2 and Lemma 2.4 were derived in Khatri [17]. In the expressions (17) and (18), $\Gamma_m(a,\rho)$ and $\Gamma_m(a,-\rho)$, for an ordered partition ρ of r, $\rho = (r_1,\dots,r_m)$, are defined by

$$\Gamma_m(a,\rho) = (a)_\rho \Gamma_m(a), \quad \Gamma_m(a,0) = \Gamma_m(a)$$

and

$$\Gamma_m(a,-\rho) = \frac{(-1)^r\, \Gamma_m(a)}{(-a+(m+1)/2)_\rho}, \quad \operatorname{Re}(a) > r_1 + \frac{m-1}{2},$$

respectively.

Definition 2.1. *The extended matrix variate gamma function, denoted by* $\Gamma_m(a;\Sigma)$*, is defined by*

$$\Gamma_m(a;\Sigma) = \int_{Z>0} \det(Z)^{a-(m+1)/2} \operatorname{etr}(-Z - \Sigma Z^{-1})\, \mathrm{d}Z, \tag{21}$$

where $\operatorname{Re}(\Sigma) > 0$ *and* a *is an arbitrary complex number.*

From (21), one can easily see that for $\operatorname{Re}(\Sigma) > 0$ and $H \in O(m)$, $\Gamma_m(a; H\Sigma H') = \Gamma_m(a;\Sigma)$ thereby $\Gamma_m(a;\Sigma)$ depends on the matrix Σ only through its eigenvalues if Σ is a real matrix.

From the definition, it is clear that if $\Sigma = 0$, then for $\operatorname{Re}(a) > (m-1)/2$, the extended matrix variate gamma function reduces to the multivariate gamma function $\Gamma_m(a)$.

Definition 2.2. *The extended matrix variate beta function, denoted by* $B_m(a,b;\Sigma)$*, is defined as*

$$B_m(a,b;\Sigma) = \int_0^{I_m} \operatorname{etr}[-\Sigma Z^{-1}(I_m - Z)^{-1}] \times \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2}\, \mathrm{d}Z, \tag{22}$$

where where $\operatorname{Re}(\Sigma) > 0$ *and* a *and* b *are arbitrary complex numbers. If* $\Sigma = 0$*, then* $\operatorname{Re}(a) > (m-1)/2$ *and* $\operatorname{Re}(b) > (m-1)/2$.

3 Generalized extended matrix variate Gamma function

A matrix variate generalization of the generalized extended gamma function can be defined in the following way:

Definition 3.1. *The generalized extended matrix variate gamma function, denoted by* $\Gamma_m^{(\alpha,\beta)}(a;\Sigma)$*, is defined by*

$$\Gamma_m^{(\alpha,\beta)}(a;\Sigma) = \int_{Z>0} \det(Z)^{a-(m+1)/2}\Phi(\alpha;\beta;-Z-Z^{-1/2}\Sigma Z^{-1/2})\,\mathrm{d}Z, \quad (23)$$

where $\Sigma > 0$ *and* a *is an arbitrary complex number.*

From the definition it is clear that for $\alpha = \beta$, the generalized extended matrix variate gamma function reduces to an extended matrix variate gamma function, *i.e.*, $\Gamma_m^{(\alpha,\alpha)}(a;\Sigma) = \Gamma_m(a;\Sigma)$. Further, if $\alpha = \beta$ and $\Sigma = 0$, then for $\mathrm{Re}(a) > (m-1)/2$, the generalized extended matrix variate gamma function reduces to the multivariate gamma function $\Gamma_m(a)$.

Replacing $\Phi(\alpha;\beta;-Z-Z^{-1/2}\Sigma Z^{-1/2})$ by its integral representation, namely,

$$\begin{aligned}\Phi(\alpha;\beta;&-Z-Z^{-1/2}\Sigma Z^{-1/2})\\ &= \frac{1}{B_m(\alpha,\beta-\alpha)}\int_0^{I_m} \mathrm{etr}[-(Z+Z^{-1/2}\Sigma Z^{-1/2})Y]\\ &\quad\times \det(Y)^{\alpha-(m+1)/2}\det(I_m-Y)^{\beta-\alpha-(m+1)/2}\,\mathrm{d}Y, \qquad (24)\end{aligned}$$

where $\mathrm{Re}(\alpha) > (m-1)/2$ and $\mathrm{Re}(\beta-\alpha) > (m-1)/2$, in (23), an alternative integral representation of the generalized extended matrix variate gamma function can be given as

$$\begin{aligned}\Gamma_m^{(\alpha,\beta)}(a;\Sigma) &= \frac{1}{B_m(\alpha,\beta-\alpha)}\int_0^{I_m}\det(Y)^{\alpha-(m+1)/2}\det(I_m-Y)^{\beta-\alpha-(m+1)/2}\\ &\quad\times\int_{Z>0}\det(Z)^{a-(m+1)/2}\,\mathrm{etr}[-(Z+Z^{-1/2}\Sigma Z^{-1/2})Y]\,\mathrm{d}Z\,\mathrm{d}Y.\end{aligned} \quad (25)$$

Two special cases of (25) are worth mentioning. For $m = 1$, this expression simplifies to

$$\Gamma^{(\alpha,\beta)}(a;\sigma) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 y^{\alpha-a-1}(1-y)^{\beta-\alpha-1}\Gamma(a;\sigma y^2)\,\mathrm{d}y.$$

Further, for $\Sigma = I_m$, substituting $X = Y^{1/2}ZY^{1/2}$ with the Jacobian $J(Z \to X) = \det(Y)^{-(m+1)/2}$ in (25) and applying (21), the expression for $\Gamma_m^{(\alpha,\beta)}(a; I_m)$ is derived as

$$\Gamma_m^{(\alpha,\beta)}(a; I_m) = \frac{1}{B_m(\alpha, \beta-\alpha)} \int_0^{I_m} \det(Y)^{\alpha-a-(m+1)/2} \times \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \Gamma_m(a; Y^2)\,\mathrm{d}Y. \tag{26}$$

In the following theorem we establish a relationship between generalized extended gamma function of matrix argument and multivariate gamma function through an integral involving the generalized extended gamma function of matrix argument and zonal polynomials.

Theorem 3.1. *For* $\mathrm{Re}(s) > (m-1)/2$, $\mathrm{Re}(s+a) > (m-1)/2$, $\mathrm{Re}(\alpha - a - 2s) > (m-1)/2$ *and* $\mathrm{Re}(\beta - a - 2s) > (m-1)/2$, *we have*

$$\begin{aligned}\int_{\Sigma>0} &\det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma) \Gamma_m^{(\alpha,\beta)}(a;\Sigma)\,\mathrm{d}\Sigma \\ &= \frac{\Gamma_m(s,\kappa)\Gamma_m(a+s,\kappa)}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(Y)^{\alpha-a-2s-(m+1)/2} \\ &\quad \times \det(I_m - Y)^{\beta-\alpha-(m+1)/2} C_\kappa(Y^{-2})\,\mathrm{d}Y. \end{aligned} \tag{27}$$

Proof. Replacing $\Gamma_m^{(\alpha,\beta)}(a;\Sigma)$ by its integral representation given in (25) and changing the order of integration, the left hand side integral in (27) is re-written as

$$\begin{aligned}&\frac{1}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(Y)^{\alpha-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\ &\times \int_{Z>0} \mathrm{etr}(-ZY) \det(Z)^{a-(m+1)/2} \\ &\times \int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma)\,\mathrm{etr}(-Z^{-1/2}\Sigma Z^{-1/2} Y)\,\mathrm{d}\Sigma\,\mathrm{d}Z\,\mathrm{d}Y, \end{aligned} \tag{28}$$

where $\mathrm{Re}(\alpha) > (m-1)/2$ and $\mathrm{Re}(\beta-\alpha) > (m-1)/2$.

Further, using Lemma 2.1, the integral involving Σ is evaluated as

$$\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma)\,\mathrm{etr}(-Z^{-1/2}\Sigma Z^{-1/2} Y)\,\mathrm{d}\Sigma$$

$$= \det(Z)^s \det(Y)^{-s}\Gamma_m(s,\kappa)C_\kappa(ZY^{-1}), \quad \operatorname{Re}(s) > \frac{m-1}{2}. \tag{29}$$

Replacing (29) in (28), to obtain

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma)\Gamma_m^{(\alpha,\beta)}(a;\Sigma)\,\mathrm{d}\Sigma \\
&= \frac{\Gamma_m(s,\kappa)}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(Y)^{\alpha-s-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\
&\quad \times \int_{Z>0} \operatorname{etr}(-ZY) \det(Z)^{a+s-(m+1)/2} C_\kappa(Y^{-1}Z)\,\mathrm{d}Z\,\mathrm{d}Y.
\end{aligned} \tag{30}$$

Now, integration of Z using Lemma 2.1 yields the desired result. □

Theorem 3.2. *For a symmetric positive definite matrix T of order m, $\operatorname{Re}(s) > (m-1)/2$, $\operatorname{Re}(s+a) > (m-1)/2$, $\operatorname{Re}(\alpha-a-2s) > (m-1)/2$ and $\operatorname{Re}(\beta-a-2s) > (m-1)/2$, we have*

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(T\Sigma)\Gamma_m^{(\alpha,\beta)}(a;\Sigma)\,\mathrm{d}\Sigma \\
&= \frac{\Gamma_m(s,\kappa)\Gamma_m(a+s,\kappa)}{B_m(\alpha,\beta-\alpha)} \frac{C_\kappa(T)}{C_\kappa(I_m)} \int_0^{I_m} \det(Y)^{\alpha-a-2s-(m+1)/2} \\
&\quad \times \det(I_m - Y)^{\beta-\alpha-(m+1)/2} C_\kappa(Y^{-2})\,\mathrm{d}Y.
\end{aligned} \tag{31}$$

Corollary 3.2.1. *For $\operatorname{Re}(s) > (m-1)/2$, $\operatorname{Re}(s+a) > (m-1)/2$, $\operatorname{Re}(\alpha - a - 2s) > (m-1)/2$ and $\operatorname{Re}(\beta-a-2s) > (m-1)/2$, we have*

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} \Gamma_m^{(\alpha,\beta)}(a;\Sigma)\,\mathrm{d}\Sigma \\
&= \frac{\Gamma_m(s)\Gamma_m(a+s)\Gamma_m(\beta)\Gamma_m(\alpha-a-2s)}{\Gamma_m(\alpha)\Gamma_m(\beta-a-2s)}.
\end{aligned} \tag{32}$$

Note that the above corollary gives an interesting relationship between the generalized extended gamma function of matrix argument and multivariate gamma function. Substituting $s = (m+1)/2$, in (32), we obtain

$$\begin{aligned}
&\int_{\Sigma>0} \Gamma_m^{(\alpha,\beta)}(a;\Sigma)\,\mathrm{d}\Sigma \\
&= \frac{\Gamma_m[(m+1)/2]\Gamma_m[a+(m+1)/2]\Gamma_m(\beta)\Gamma_m(\alpha-a-m-1)}{\Gamma_m(\alpha)\Gamma_m(\beta-a-m-1)}.
\end{aligned}$$

Theorem 3.3. *For* $\operatorname{Re}(s) > k_1+(m-1)/2$ *and* $\operatorname{Re}(s+a) > k_1+(m-1)/2$,

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma^{-1}) \Gamma_m^{(\alpha,\beta)}(a;\Sigma)\, \mathrm{d}\Sigma \\
&= \frac{\Gamma_m(s,-\kappa)\Gamma_m(a+s,-\kappa)}{B_m(\alpha,\beta-\alpha)} \\
&\quad \times \int_0^{I_m} \det(Y)^{\alpha-a-2s-(m+1)/2} \det(I_m-Y)^{\beta-\alpha-(m+1)/2} C_\kappa(Y^2)\, \mathrm{d}Y \\
&= \frac{\Gamma_m(s)\Gamma_m(a+s)}{B_m(\alpha,\beta-\alpha)(-s+(m+1)/2)_\kappa(-a-s+(m+1)/2)_\kappa} \\
&\quad \times \int_0^{I_m} \det(Y)^{\alpha-a-2s-(m+1)/2} \det(I_m-Y)^{\beta-\alpha-(m+1)/2} C_\kappa(Y^2)\, \mathrm{d}Y, \qquad (33)
\end{aligned}$$

where $\kappa = (k_1,\ldots,k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$ *and* $k_1+\cdots+k_m = k$.

Proof. Replacing $\Gamma_m^{(\alpha,\beta)}(a;\Sigma)$ by its integral representation given in (25), the left hand side integral in (33) is re-written as

$$\begin{aligned}
&\frac{1}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(Y)^{\alpha-(m+1)/2} \det(I_m-Y)^{\beta-\alpha-(m+1)/2} \\
&\quad \times \int_{Z>0} \operatorname{etr}(-ZY) \det(Z)^{a-(m+1)/2} \\
&\quad \times \int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma^{-1}) \operatorname{etr}(-Z^{-1/2}\Sigma Z^{-1/2} Y)\mathrm{d}\Sigma\, \mathrm{d}Z\, \mathrm{d}Y. \qquad (34)
\end{aligned}$$

Now, integrating first with respect to Σ and then with respect to Z by using Lemma 2.2, we obtain the desired result. □

Theorem 3.4. *For* $\Sigma > 0$ *and* $a > (m-1)/2$, $\alpha - a > (m-1)/2$ *and* $\beta - a > (m-1)/2$, *we have*

$$\Gamma_m^{(\alpha,\beta)}(a;\Sigma) \leq \frac{\Gamma_m(a)\Gamma_m(\beta)\Gamma_m(\alpha-a)}{\Gamma_m(\alpha)\Gamma_m(\beta-a)}.$$

Proof. Let Z, Y and Σ be symmetric positive definite matrices of order m. Further, let $\lambda_1,\ldots,\lambda_m$ be the characteristic roots of the matrix $Y^{1/2}Z^{-1/2}\Sigma Z^{-1/2}Y^{1/2}$. Then

$$\operatorname{etr}(-Z^{-1/2}\Sigma Z^{-1/2}Y) = \exp[-(\lambda_1+\cdots+\lambda_m)].$$

Since $Z > 0$, $Y > 0$ and $\Sigma > 0$, we have $Y^{1/2}Z^{-1/2}\Sigma Z^{-1/2}Y^{1/2} > 0$, and therefore $\lambda_1 + \cdots + \lambda_m > 0$. Further, as $\exp(-t) < 1$, for all $t > 0$, we have

$$\operatorname{etr}(-Z^{-1/2}\Sigma Z^{-1/2}Y) = \exp[-(\lambda_1 + \cdots + \lambda_m)] < 1. \tag{35}$$

Now, using the above inequality in (25), we have

$$\begin{aligned}\Gamma_m^{(\alpha,\beta)}(a;\Sigma) \le & \frac{1}{B_m(\alpha,\beta-\alpha)} \\ & \times \int_0^{I_m} \det(Y)^{\alpha-(m+1)/2} \det(I_m - Y)^{\beta-\alpha-(m+1)/2} \\ & \times \int_{Z>0} \det(Z)^{a-(m+1)/2} \operatorname{etr}(-ZY)\, \mathrm{d}Z\, \mathrm{d}Y. \end{aligned} \tag{36}$$

Finally, integrating Z and Y by using multivariate gamma and multivariate beta integrals and simplifying the resulting expression, we obtain the desired result. □

Theorem 3.5. *Suppose that σ_1 and σ_n are the smallest and largest eigenvalues of the matrix $\Sigma > 0$. Then*

$$\Gamma_m^{(\alpha,\beta)}(a;\sigma_n I_m) \le \Gamma_m^{(\alpha,\beta)}(a;\Sigma) \le \Gamma_m^{(\alpha,\beta)}(a;\sigma_1 I_m).$$

Proof. Note that

$$\sigma_1 \operatorname{tr}(Z^{-1/2}YZ^{-1/2}) \le \operatorname{tr}(\Sigma Z^{-1/2}YZ^{-1/2}) \le \sigma_n \operatorname{tr}(Z^{-1/2}YZ^{-1/2})$$

and therefore

$$\begin{aligned}\exp[-\sigma_n(Z^{-1/2}YZ^{-1/2})] &\le \operatorname{etr}(-Z^{-1/2}\Sigma Z^{-1/2}Y) \\ &\le \exp[-\sigma_1(Z^{-1/2}YZ^{-1/2})].\end{aligned}$$

Now, applying the above inequality in (25), and using the integral representation of the generalized extended gamma function given in (25), we obtain the desired result. □

By Hölder's inequality, it is possible to obtain an interesting inequality that follows.

Theorem 3.6. *Let $1 < p < \infty$ and $(1/p) + (1/q) = 1$. Then, for $\Sigma \geq 0$, $x > (m-1)/2$ and $y > (m-1)/2$, we have*

$$\Gamma_m^{(\alpha,\beta)}\left(\frac{x}{p} + \frac{y}{q}; \Sigma\right) \leq \left[\Gamma_m^{(\alpha,\beta)}(x; \Sigma)\right]^{1/p} \left[\Gamma_m^{(\alpha,\beta)}(y; \Sigma)\right]^{1/q}. \tag{37}$$

Proof. Substituting $a = x/p + y/q$ in (23) and using Hölder's inequality, we obtain the desired result. □

Substituting $p = q = 2$ in (37), we obtain

$$\Gamma_m^{(\alpha,\beta)}\left(\frac{x+y}{2}; \Sigma\right) \leq \sqrt{\Gamma_m^{(\alpha,\beta)}(x; \Sigma)\Gamma_m^{(\alpha,\beta)}(y; \Sigma)},$$

where $x > (m-1)/2)$, $y > (m-1)/2$, and $\Sigma \geq 0$.

4 Generalized extended matrix variate Beta function

In this section, a matrix variate generalization of (6) is defined and several of its properties are studied.

Definition 4.1. *The generalized extended matrix variate beta function, denoted by $B_m^{(\alpha,\beta)}(a, b; \Sigma)$, is defined as*

$$\begin{aligned} B_m^{(\alpha,\beta)}(a, b; \Sigma) = \int_0^{I_m} & \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} \\ & \times \Phi(\alpha; \beta; -\Sigma Z^{-1}(I_m - Z)^{-1}) \, \mathrm{d}Z, \end{aligned} \tag{38}$$

where a and b are arbitrary complex numbers and $\Sigma > 0$. If $\Sigma = 0$, then $\mathrm{Re}(a) > (m-1)/2$, $\mathrm{Re}(b) > (m-1)/2$.

Using Kummer's relation (16), the above expression can also be written as

$$\begin{aligned} B_m^{(\alpha,\beta)}(a, b; \Sigma) = \int_0^{I_m} & \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} \\ & \times \mathrm{etr}[-\Sigma Z^{-1}(I_m - Z)^{-1}] \end{aligned}$$

$$\times \Phi(\beta-\alpha;\beta;\Sigma Z^{-1}(I_m-Z)^{-1})\,\mathrm{d}Z. \tag{39}$$

From (38), it is apparent that $B_m^{(\alpha,\beta)}(a,b;\Sigma)=B_m^{(\alpha,\beta)}(b,a;\Sigma)$. Further, $B_m^{(\alpha,\beta)}(a,b;\Sigma)=B_m^{(\alpha,\beta)}(a,b;H\Sigma H')$, $H\in O(m)$, thereby $B_m^{(\alpha,\beta)}(a,b;\Sigma)$ is a function of the eigenvalues of the matrix $\Sigma>0$.

Replacing the confluent hypergeometric function by its integral representation in (38), changing the order of integration, and integrating Z by using (22), we obtain

$$\begin{aligned} B_m^{(\alpha,\beta)}(a,b;\Sigma) = {} & \frac{1}{B_m(\alpha,\beta-\alpha)} \\ & \times \int_0^{I_m} \det(X)^{\alpha-(m+1)/2}\det(I_m-X)^{\beta-\alpha-(m+1)/2} \\ & \times B_m(a,b;\Sigma^{1/2}X\Sigma^{1/2})\,\mathrm{d}X. \end{aligned} \tag{40}$$

Theorem 4.1. *For* $\mathrm{Re}(s)>(m-1)/2$, $\mathrm{Re}(s+a)>(m-1)/2$, $\mathrm{Re}(s+b)>(m-1)/2$, $\mathrm{Re}(\alpha-s)>(m-1)/2$ *and* $\mathrm{Re}(\beta-s)>(m-1)/2$,

$$\begin{aligned} & \int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2}B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma \\ & \quad = \frac{\Gamma_m(s)\Gamma_m(\beta)\Gamma_m(\alpha-s)}{\Gamma_m(\alpha)\Gamma_m(\beta-s)}B_m\left(a+s,b+s\right). \end{aligned} \tag{41}$$

Proof. Replacing $B_m^{(\alpha,\beta)}(a,b;\Sigma)$ by its equivalent integral representation given in (39) and changing the order of integration, the integral in (41) is rewritten as

$$\begin{aligned} & \int_0^{I_m}\det(Z)^{a-(m+1)/2}\det(I_m-Z)^{b-(m+1)/2}\int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2} \\ & \qquad \mathrm{etr}[-\Sigma Z^{-1}(I_m-Z)^{-1}]\Phi\left(\beta-\alpha;\beta;\Sigma Z^{-1}(I_m-Z)^{-1}\right)\mathrm{d}\Sigma\,\mathrm{d}Z \\ & \quad = \frac{\Gamma_m(s)\Gamma_m(\beta)\Gamma_m(\alpha-s)}{\Gamma_m(\alpha)\Gamma_m(\beta-s)} \\ & \qquad \times \int_0^{I_m}\det(Z)^{a+s-(m+1)/2}\det(I_m-Z)^{b+s-(m+1)/2}\,\mathrm{d}Z, \end{aligned} \tag{42}$$

where we have used the result

$$\int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2}\,\mathrm{etr}[-\Sigma Z^{-1}(I_m-Z)^{-1}]$$

$$\begin{aligned}
&\times \Phi\left(\beta-\alpha;\beta;\Sigma Z^{-1}(I_m-Z)^{-1}\right)\,\mathrm{d}\Sigma\\
&=\Gamma_m(s)\det(Z)^s\det(I_m-Z)^s{}_2F_1(s,\beta-\alpha;\beta;I_m)\\
&=\Gamma_m(s)\det(Z)^s\det(I_m-Z)^s\frac{\Gamma_m(\beta)\Gamma_m(\alpha-s)}{\Gamma_m(\alpha)\Gamma_m(\beta-s)}.
\end{aligned}$$

Finally, evaluating (42) by using the definition of multivariate beta function, we obtain the desired result. □

By letting $s=(m+1)/2$, in (41), we obtain an interesting relation

$$\begin{aligned}
\int_{\Sigma>0} B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma = &\frac{\Gamma_m[(m+1)/2]\Gamma_m(\beta)\Gamma_m[\alpha-(m+1)/2]}{\Gamma_m(\alpha)\Gamma_m[\beta-(m+1)/2]}\\
&\times B_m\left(a+\frac{m+1}{2},b+\frac{m+1}{2}\right)
\end{aligned}$$

between the multivariate beta function and the generalized extended beta function of matrix argument.

Theorem 4.2. *For a and b arbitrary complex numbers and* $\mathrm{Re}(\Sigma)>0$,

$$B_m^{(\alpha,\beta)}(a,b;\Sigma)=\int_{Y>0}\frac{\det(Y)^{a-(m+1)/2}}{\det(I_m+Y)^{a+b}}\Phi(\alpha;\beta;-\Sigma(Y+2I_m+Y^{-1}))\,\mathrm{d}Y. \tag{43}$$

Proof. Substituting $Z=(I_m+Y)^{-1}Y$ with the Jacobian $J(Z\to Y)=\det(I_m+Y)^{-(m+1)}$ in (38), we obtain the desired result. □

Theorem 4.3. *For a and b arbitrary complex numbers and* $\mathrm{Re}(\Sigma)>0$,

$$\begin{aligned}
B_m^{(\alpha,\beta)}(a,b;\Sigma)=\frac{1}{2}\int_{Y>0}&\frac{\det(Y)^{a-(m+1)/2}+\det(Y)^{b-(m+1)/2}}{\det(I_m+Y)^{a+b}}\\
&\times\Phi(\alpha;\beta;-\Sigma(Y+2I_m+Y^{-1}))\,\mathrm{d}Y.
\end{aligned}$$

Proof. Noting that

$$B_m^{(\alpha,\beta)}(a,b;\Sigma)=\frac{1}{2}[B_m^{(\alpha,\beta)}(a,b;\Sigma)+B_m^{(\alpha,\beta)}(b,a;\Sigma)]$$

and substituting for $B_m^{(\alpha,\beta)}(a,b;\Sigma)$ and $B_m^{(\alpha,\beta)}(b,a;\Sigma)$ from (43), we obtain the desired result. □

In the following theorem, we present an important inequality that shows how the extended matrix variate beta function decreases exponentially compared to the multivariate beta function.

Theorem 4.4. *For $\Sigma > 0$, $\mathrm{Re}(\alpha) > (m+1)/2$, $\mathrm{Re}(\beta-\alpha) > (m-1)/2$, $a > (m-1)/2$ and $b > (m-1)/2$,*

$$\frac{|B_m^{(\alpha,\beta)}(a,b;\Sigma)|}{B_m(a,b)} \le \Phi(\alpha;\beta;-4\Sigma) \le \frac{\exp(-m)}{\det(4\Sigma)} \frac{B_m(\alpha-1,\beta-\alpha)}{B_m(\alpha,\beta-\alpha)}.$$

Proof. For $\Sigma > 0$, $a > (m-1)/2$ and $b > (m-1)/2$, Nagar, Roldán and Gupta [20] have shown that

$$|B_m(a,b;\Sigma)| \le \mathrm{etr}(-4\Sigma) B_m(a,b) \le \frac{\exp(-m)}{\det(4\Sigma)} B_m(a,b).$$

Now, using (40) and the above inequality, we obtain

$$\begin{aligned} |B_m^{(\alpha,\beta)}(a,b;\Sigma)| &= \frac{1}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(X)^{\alpha-(m+1)/2} \\ &\quad \times \det(I_m - X)^{\beta-\alpha-(m+1)/2} |B_m(a,b;\Sigma^{1/2} X \Sigma^{1/2})| \,\mathrm{d}X \\ &\le \frac{B_m(a,b)}{B_m(\alpha,\beta-\alpha)} \int_0^{I_m} \det(X)^{\alpha-(m+1)/2} \\ &\quad \times \det(I_m - X)^{\beta-\alpha-(m+1)/2}\, \mathrm{etr}(-4\Sigma X) \,\mathrm{d}X \\ &\le \frac{B_m(a,b)}{B_m(\alpha,\beta-\alpha)} \exp(-m) \int_0^{I_m} \det(X)^{\alpha-(m+1)/2} \\ &\quad \det(I_m - X)^{\beta-\alpha-(m+1)/2} \det(4\Sigma X)^{-1} \,\mathrm{d}X. \end{aligned}$$

The desired result is now obtained by evaluating integrals using (14) and (9) and simplifying the resulting expression. □

Theorem 4.5. *Suppose that σ_1 and σ_n are the smallest and largest eigenvalues of the matrix Σ. Then*

$$B_m^{(\alpha,\beta)}(a,b;\sigma_n I_m) \le B_m^{(\alpha,\beta)}(a,b;\Sigma) \le B_m^{(\alpha,\beta)}(a,b;\sigma_1 I_m).$$

Proof. Similar to the proof of Theorem 3.5. □

The next result is obtained by applying the Minkowski inequality for determinants. The famous Minkowski inequality states that if A and B are symmetric positive definite matrices of order m, then

$$\det(A+B)^{1/m} \geq \det(A)^{1/m} + \det(B)^{1/m}.$$

Theorem 4.6. *For the generalized extended beta function of matrix argument, we have*

$$B_m^{(\alpha,\beta)}\left(a+\frac{1}{m}, b; \Sigma\right) + B_m^{(\alpha,\beta)}\left(a, b+\frac{1}{m}; \Sigma\right) \leq B_m^{(\alpha,\beta)}\left(a, b; \Sigma\right).$$

Proof. Replacing $B_m^{(\alpha,\beta)}(a+1/m, b; \Sigma)$ and $B_m^{(\alpha,\beta)}(a, b+1/m; \Sigma)$ by their respective integral representation, one obtains

$$\begin{aligned}
&B_m^{(\alpha,\beta)}\left(a+\frac{1}{m}, b; \Sigma\right) + B_m^{(\alpha,\beta)}\left(a, b+\frac{1}{m}; \Sigma\right) \\
&= \int_0^{I_m} \left[\det(Z)^{1/m} + \det(I_m - Z)^{1/m}\right] \det(Z)^{a-(m+1)/2} \\
&\qquad \times \det(I_m - Z)^{b-(m+1)/2} \Phi(\alpha; \beta; -\Sigma Z^{-1}(I_m - Z)^{-1})\, \mathrm{d}Z.
\end{aligned}$$

Now, by noting that $\det(Z)^{1/m} + \det(I_m - Z)^{1/m} \leq 1$ we obtain the desired result. $\square$

5 Results involving zonal polynomials

In this section, we will compute the integral

$$\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma) B_m^{(\alpha,\beta)}(a, b; \Sigma)\, \mathrm{d}\Sigma, \tag{44}$$

where $\mathrm{Re}(s) > (m-1)/2$, $\mathrm{Re}(s+a) > (m-1)/2$ and $\mathrm{Re}(s+b) > (m-1)/2$.

The calculation of this integral requires evaluation of the integrals of the form

$$\int_0^{I_m} \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} C_\kappa(Z(I_m - Z))\, \mathrm{d}Z, \tag{45}$$

where $\mathrm{Re}(a) > (m-1)/2$ and $\mathrm{Re}(b) > (m-1)/2$.

Recently, Nagar, Roldán-Correa and Gupta [20] have given computable representations of (45) for $k=1$ and $k=2$ which we state in the following three lemmas.

Lemma 5.1. *For* $\mathrm{Re}(a) > (m-1)/2$ *and* $\mathrm{Re}(b) > (m-1)/2$*,*

$$\int_0^{I_m} \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} C_{(1)}(Z(I_m - Z))\, \mathrm{d}Z \\ = m B_m(a,b) K_{(1)}(a,b), \tag{46}$$

where

$$K_{(1)}(a,b) = \frac{a}{a+b}\left[1 - \frac{(a+1)(m+2)}{3(a+b+1)} + \frac{(a-1/2)(m-1)}{3(a+b-1/2)}\right].$$

Proof. See Nagar, Roldán-Correa and Gupta [20]. □

Lemma 5.2. *For* $\mathrm{Re}(a) > (m-1)/2$ *and* $\mathrm{Re}(b) > (m-1)/2$*,*

$$\int_0^{I_m} \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} C_{(2)}(Z(I_m - Z))\, \mathrm{d}Z \\ = m B_m(a,b) K_{(2)}(a,b), \tag{47}$$

where

$$\begin{aligned} K_{(2)}(a,b) = \frac{(m+2)a(a+1)}{3(a+b)(a+b+1)}\Bigg[1 &- \frac{2(a+2)(m+4)}{5(a+b+2)} \\ &+ \frac{2(a-1/2)(m-1)}{5(a+b-1/2)} + \frac{(a+2)(a+3)(m+4)(m+6)}{35(a+b+2)(a+b+3)} \\ &+ \frac{(a^2-1/4)(m^2-1)}{15[(a+b)^2-1/4]} - \frac{2(a+2)(a-1/2)(m-1)(m+4)}{21(a+b+2)(a+b-1/2)}\Bigg]. \end{aligned}$$

Proof. See Nagar, Roldán-Correa and Gupta [20]. □

Lemma 5.3. *For* $\mathrm{Re}(a) > (m-1)/2$ *and* $\mathrm{Re}(b) > (m-1)/2$*,*

$$\int_0^{I_m} \det(Z)^{a-(m+1)/2} \det(I_m - Z)^{b-(m+1)/2} C_{(1^2)}(Z(I_m - Z))\, \mathrm{d}Z$$

$$= mB_m(a,b)K_{(1^2)}(a,b), \tag{48}$$

where

$$\begin{aligned}K_{(1^2)}(a,b) = \frac{(m-1)a(a-1/2)}{3(a+b)(a+b-1/2)}\Bigg[2 - \frac{(a+1)(m+2)}{a+b+1} \\ + \frac{(a-1)(m-2)}{a+b-1} + \frac{(a+1)(a+1/2)(m+1)(m+2)}{6(a+b+1)(a+b+1/2)} \\ - \frac{4(a+1)(a-1)(m^2-4)}{15(a+b+1)(a+b-1)} \\ + \frac{(a-1)(a-3/2)(m-2)(m-3)}{10(a+b-1)(a+b-3/2)}\Bigg].\end{aligned}$$

Proof. See Nagar, Roldán-Correa and Gupta [20]. □

In the following theorem and three corollaries we give closed form representations of the integral (44) for $k=1$ and $k=2$.

Theorem 5.1. *For* $\mathrm{Re}(s) > (m-1)/2$, $\mathrm{Re}(s+a) > (m-1)/2$, $\mathrm{Re}(s+b) > (m-1)/2$, $\mathrm{Re}(\alpha - s) > k_1 + (m-1)/2$ *and* $\mathrm{Re}(\beta - s) > k_1 + (m-1)/2$,

$$\begin{aligned}&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma) B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma \\ &= \frac{\Gamma_m(s)\Gamma_m(\alpha-s)\Gamma_m(\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta-s)} \frac{(s)_\kappa(-\beta+s+(m+1)/2)_\kappa}{(-\alpha+s+(m+1)/2)_\kappa} \\ &\quad\times \int_0^{I_m} \det(Z)^{a+s-(m+1)/2}\det(I_m-Z)^{b+s-(m+1)/2} C_\kappa(Z(I_m-Z))\,\mathrm{d}Z.\end{aligned} \tag{49}$$

Proof. Replacing $B_m^{(\alpha,\beta)}(a,b;\Sigma)$ by its equivalent integral representation given in (40) and changing the order of integration, the integral in (49) is rewritten as

$$\begin{aligned}&\frac{1}{B_m(\alpha,\beta-\alpha)}\int_0^{I_m} \det(Z)^{a-(m+1)/2}\det(I_m-Z)^{b-(m+1)/2} \\ &\times \int_0^{I_m}\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma)\,\mathrm{etr}[-(Z-Z^2)^{-1/2}\Sigma(Z-Z^2)^{-1/2}X]\mathrm{d}\Sigma\end{aligned}$$

$$\times \det(X)^{\alpha-(m+1)/2} \det(I_m - X)^{\beta-\alpha-(m+1)/2} \, \mathrm{d}X \, \mathrm{d}Z. \tag{50}$$

Now, evaluating the integral containing Σ using Lemma 2.1, we obtain

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_\kappa(\Sigma) \operatorname{etr}[-(Z-Z^2)^{-1/2}\Sigma(Z-Z^2)^{-1/2}X] \, \mathrm{d}\Sigma \\
&= \Gamma_m(s)(s)_\kappa \det(X)^{-s} \det(Z)^s \det(I_m - Z)^s C_\kappa(Z(I_m - Z)X^{-1}).
\end{aligned} \tag{51}$$

Further, substituting (51) in (50) and integrating with respect to X by using Lemma 2.4, we obtain

$$\begin{aligned}
&\frac{1}{B_m(\alpha, \beta-\alpha)} \int_0^{I_m} \det(X)^{\alpha-s-(m+1)/2} \det(I_m - X)^{\beta-\alpha-(m+1)/2} \\
&\qquad \times C_\kappa(Z(I_m - Z)X^{-1}) \, \mathrm{d}X \\
&= \frac{\Gamma_m(\alpha - s)\Gamma_m(\beta)}{\Gamma_m(\alpha)\Gamma_m(\beta - s)} \frac{(-\beta + s + (m+1)/2)_\kappa}{(-\alpha + s + (m+1)/2)_\kappa} C_\kappa(Z(I_m - Z)),
\end{aligned} \tag{52}$$

where $\mathrm{Re}(\alpha - s) > k_1 + (m-1)/2$ and $\mathrm{Re}(\beta - s) > k_1 + (m-1)/2$. Now, substituting appropriately, we obtain the result. □

Corollary 5.1.1. *For* $\mathrm{Re}(s) > (m-1)/2$, $\mathrm{Re}(s+a) > (m-1)/2$ *and* $\mathrm{Re}(s+b) > (m-1)/2$,

$$\begin{aligned}
&\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_{(1)}(\Sigma) B_m^{(\alpha,\beta)}(a, b; \Sigma) \, \mathrm{d}\Sigma \\
&\qquad = sm \frac{\Gamma_m(s)\Gamma_m(\alpha - s)\Gamma_m(\beta)}{\Gamma_m(\beta - s)\Gamma_m(\alpha)} \frac{(-\beta + s + (m+1)/2)}{(-\alpha + s + (m+1)/2)} \\
&\qquad\quad \times B_m(a+s, b+s) K_{(1)}(a+s, b+s),
\end{aligned} \tag{53}$$

where $\mathrm{Re}(\alpha - s) > (m+1)/2$ *and* $\mathrm{Re}(\beta - s) > (m+1)/2$.

Proof. Substituting $\kappa = (1)$ in (49) and using Lemma 5.1, we obtain the desired result. □

Corollary 5.1.2. *For* $\mathrm{Re}(s) > (m-1)/2$, $\mathrm{Re}(s+a) > (m-1)/2$ *and* $\mathrm{Re}(s+b) > (m-1)/2$,

$$\int_{\Sigma>0} \det(\Sigma)^{s-(m+1)/2} C_{(2)}(\Sigma) B_m^{(\alpha,\beta)}(a, b; \Sigma) \, \mathrm{d}\Sigma$$

$$
\begin{aligned}
&= s(s+1)m\frac{\Gamma_m(s)\Gamma_m(\alpha-s)\Gamma_m(\beta)}{\Gamma_m(\beta-s)\Gamma_m(\alpha)}\\
&\quad\times\frac{(-\beta+s+(m+1)/2)(-\beta+s+(m+3)/2)}{(-\alpha+s+(m+1)/2)(-\alpha+s+(m+3)/2)}\\
&\quad\times B_m(a+s,b+s)K_{(2)}(a+s,b+s),
\end{aligned}
\tag{54}
$$

where $\operatorname{Re}(\alpha-s)>(m+3)/2$ *and* $\operatorname{Re}(\beta-s)>(m+3)/2$.

Proof. Substituting $\kappa=(2)$ in (49) and using Lemma 5.2, we obtain the desired result. ☐

Corollary 5.1.3. *For* $\operatorname{Re}(s)>(m-1)/2$, $\operatorname{Re}(s+a)>(m-1)/2$ *and* $\operatorname{Re}(s+b)>(m-1)/2$,

$$
\begin{aligned}
&\int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2}C_{(1^2)}(\Sigma)B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma\\
&= s\left(s-\frac{1}{2}\right)m\frac{\Gamma_m(s)\Gamma_m(\alpha-s)\Gamma_m(\beta)}{\Gamma_m(\beta-s)\Gamma_m(\alpha)}\\
&\quad\times\frac{(-\beta+s+(m+1)/2)(-\beta+s+m/2)}{(-\alpha+s+(m+1)/2)(-\alpha+s+m/2)}\\
&\quad\times B_m(a+s,b+s)K_{(1^2)}(a+s,b+s),
\end{aligned}
\tag{55}
$$

where $\operatorname{Re}(\alpha-s)>(m+1)/2$ *and* $\operatorname{Re}(\beta-s)>(m+1)/2$.

Proof. Substituting $\kappa=(1,1)$ in (49) and using Lemma 5.3, we obtain the desired result. ☐

Corollary 5.1.4. *For* $\operatorname{Re}(s)>(m-1)/2$, $\operatorname{Re}(s+a)>(m-1)/2$ *and* $\operatorname{Re}(s+b)>(m-1)/2$, $\operatorname{Re}(\alpha-s)>(m+3)/2$ *and* $\operatorname{Re}(\beta-s)>(m+3)/2$,

$$
\begin{aligned}
&\int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2}(\operatorname{tr}\Sigma)^2B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma\\
&= sm\frac{\Gamma_m(s)\Gamma_m(\alpha-s)\Gamma_m(\beta)}{\Gamma_m(\beta-s)\Gamma_m(\alpha)}\frac{(-\beta+s+(m+1)/2)}{(-\alpha+s+(m+1)/2)}B_m(a+s,b+s)\\
&\quad\times\left[(s+1)\frac{(-\beta+s+(m+3)/2)}{(-\alpha+s+(m+3)/2)}K_{(2)}(a+s,b+s)\right.\\
&\qquad\left.+\left(s-\frac{1}{2}\right)\frac{(-\beta+s+m/2)}{(-\alpha+s+m/2)}K_{(1^2)}(a+s,b+s)\right].
\end{aligned}
$$

Proof. We obtain the desired result by summing (54) and (55) and using the result $C_{(2)}(\Sigma)+C_{(1^2)}(\Sigma)=(\operatorname{tr}\Sigma)^2$. □

Corollary 5.1.5. *For* $\operatorname{Re}(s)>(m-1)/2$, $\operatorname{Re}(s+a)>(m-1)/2$ *and* $\operatorname{Re}(s+b)>(m-1)/2$, $\operatorname{Re}(\alpha-s)>(m+3)/2$ *and* $\operatorname{Re}(\beta-s)>k_1+(m+3)/2$,

$$\begin{aligned}
&\int_{\Sigma>0}\det(\Sigma)^{s-(m+1)/2}(\operatorname{tr}\Sigma^2)B_m^{(\alpha,\beta)}(a,b;\Sigma)\,\mathrm{d}\Sigma\\
&= sm\frac{\Gamma_m(s)\Gamma_m(\alpha-s)\Gamma_m(\beta)}{\Gamma_m(\beta-s)\Gamma_m(\alpha)}\frac{(-\beta+s+(m+1)/2)}{(-\alpha+s+(m+1)/2)}B_m(a+s,b+s)\\
&\quad\times\left[(s+1)\frac{(-\beta+s+(m+3)/2)}{(-\alpha+s+(m+3)/2)}K_{(2)}(a+s,b+s)\right.\\
&\qquad\left.-\frac{1}{2}\left(s-\frac{1}{2}\right)\frac{(-\beta+s+m/2)}{(-\alpha+s+m/2)}K_{(1^2)}(a+s,b+s)\right].
\end{aligned}$$

Proof. We obtain the desired result by using $C_{(2)}(\Sigma)-C_{(1^2)}(\Sigma)/2=(\operatorname{tr}\Sigma^2)$. □

6 Application to multivariate statistics

The Wishart distribution, which is the distribution of the sample variance covariance matrix when sampling from a multivariate normal distribution, is an important distribution in multivariate statistival analysis. Recently, Bekker et al. [2, 3] and Bekker, Roux and Arashi [1] have used Wishart distribution in deriving a number of matrix variate distributions. Further, Bodnar, Mazur and Okhrin [5] have considered exact and approximate distribution of the product of a Wishart matrix and a Gaussian vector. The inverted Wishart distribution is widely used as a conjugate prior in Bayesian statistics (Iranmanesh et al. [14]). Knowledge of densities of functions of inverted Wishart matrices is useful for the implementation of several statistical procedures and in this regard we show that the distribution of the sum of dependent generalized inverted Wishart matrices can be written in terms generalized extended beta function of matrix argument. If $W\sim IW_m(\nu,\Psi)$, $\nu>m-1$, $\Psi>0$, then its p.d.f. is given by

$$\frac{\det(\Psi)^{(\nu-m-1)/2}\operatorname{etr}\left(-\Psi W^{-1}/2\right)\det(W)^{-\nu/2}}{2^{m(\nu-m-1)/2}\Gamma_m\left[(\nu-m-1)/2\right]},\quad W>0.$$

By replacing $\operatorname{etr}\left(-\Psi W^{-1}/2\right)$ by the confluent hypergeometric function of matrix argument $\Phi\left(\alpha;\beta;-\Psi W^{-1}/2\right)$ and evaluating the normalizing constant, a generalization of the inverted Wishart distribution can be defined by the density

$$\frac{\Gamma_m(\alpha)\Gamma_m[\beta-(\nu-m-1)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha-(\nu-m-1)/2]}\frac{2^{-m(\nu-m-1)/2}\det(\Psi)^{(\nu-m-1)/2}}{\Gamma_m\left[(\nu-m-1)/2\right]}$$
$$\times\det(W)^{-\nu/2}\Phi\left(\alpha;\beta;-\frac{1}{2}\Psi W^{-1}\right),\quad W>0.$$

Further, a bi-matrix variate generalization of the above density can be defined as

$$\frac{\Gamma_m(\alpha)\Gamma_m[\beta-(\nu_1+\nu_2-2m-2)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha-(\nu_1+\nu_2-2m-2)/2]}$$
$$\times\frac{2^{-m(\nu_1+\nu_2-2m-2)/2}\det(\Psi)^{(\nu_1+\nu_2-2m-2)/2}}{\Gamma_m\left[(\nu_1-m-1)/2\right]\Gamma_m\left[(\nu_2-m-1)/2\right]}$$
$$\times\det(W_1)^{-\nu_1/2}\det(W_2)^{-\nu_2/2}\Phi\left(\alpha;\beta;-\frac{1}{2}\Psi(W_1^{-1}+W_2^{-1})\right),\tag{56}$$

where $W_1>0$ and $W_2>0$. Note that if we take $\alpha=\beta$ in the above density, then W_1 and W_2 are independent, $W_1\sim IW_m(\nu_1,\Psi)$ and $W_2\sim IW_m(\nu_2,\Psi)$. Further, the marginal density of W_1 is given by

$$\frac{\Gamma_m[\alpha-(\nu_2-m-1)/2]\Gamma_m[\beta-(\nu_1+\nu_2-2m-2)/2]}{\Gamma_m\beta-(\nu_2-m-1)/2]\Gamma_m[\alpha-(\nu_1+\nu_2-2m-2)/2]}$$
$$\times\frac{2^{-m(\nu_1-m-1)/2}\det(\Psi)^{(\nu_1-m-1)/2}}{\Gamma_m\left[(\nu_1-m-1)/2\right]}\det(W_1)^{-\nu_1/2}$$
$$\times\Phi\left(\alpha-\frac{\nu_2-m-1}{2};\beta-\frac{\nu_2-m-1}{2};-\frac{1}{2}\Psi W_1^{-1}\right),\tag{57}$$

where $W_1>0$. Likewise, the marginal density of W_2 is given by

$$\frac{\Gamma_m[\alpha-(\nu_1-m-1)/2]\Gamma_m[\beta-(\nu_1+\nu_2-2m-2)/2]}{\Gamma_m\beta-(\nu_1-m-1)/2]\Gamma_m[\alpha-(\nu_1+\nu_2-2m-2)/2]}$$
$$\times\frac{2^{-m(\nu_2-m-1)/2}\det(\Psi)^{(\nu_2-m-1)/2}}{\Gamma_m\left[(\nu_2-m-1)/2\right]}\det(W_2)^{-\nu_2/2}$$

$$\times \Phi\left(\alpha - \frac{\nu_1 - m - 1}{2}; \beta - \frac{\nu_1 - m - 1}{2}; -\frac{1}{2}\Psi W_2^{-1}\right), \qquad (58)$$

where $W_2 > 0$.

Theorem 6.1. *Suppose that the joint density of the random matrices W_1 and W_2 is given by (56). Then, the p.d.f. of the sum $W = W_1 + W_2$ is derived as*

$$\begin{aligned}
&\frac{\Gamma_m(\alpha)\Gamma_m[\beta - (\nu_1 + \nu_2 - 2m - 2)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha - (\nu_1 + \nu_2 - 2m - 2)/2]} \\
&\quad \times \frac{2^{-m(\nu_1+\nu_2-2m-2)/2} \det(\Psi)^{(\nu_1+\nu_2-2m-2)/2}}{\Gamma_m\left[(\nu_1 - m - 1)/2\right]\Gamma_m\left[(\nu_2 - m - 1)/2\right]} \det(W)^{-(\nu_1+\nu_2-m-1)/2} \\
&\quad \times B_m^{(\alpha,\beta)}\left(-\frac{\nu_1 - m - 1}{2}, -\frac{\nu_2 - m - 1}{2}; \frac{1}{2}\Psi^{1/2}W^{-1}\Psi^{1/2}\right), \quad W > 0.
\end{aligned}$$

Proof. Transforming $W = W_1 + W_2$, $R = W^{-1/2}W_1W^{-1/2}$ with the Jacobian $J(W_1, W_2 \to R, W) = \det(W)^{(m+1)/2}$ in (56), the joint p.d.f. of R and W is obtained as

$$\begin{aligned}
&\frac{\Gamma_m(\alpha)\Gamma_m[\beta - (\nu_1 + \nu_2 - 2m - 2)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha - (\nu_1 + \nu_2 - 2m - 2)/2]} \\
&\quad \times \frac{2^{-m(\nu_1+\nu_2-2m-2)/2} \det(\Psi)^{(\nu_1+\nu_2-2m-2)/2}}{\Gamma_m\left[(\nu_1 - m - 1)/2\right]\Gamma_m\left[(\nu_2 - m - 1)/2\right]} \\
&\quad \times \det(W)^{-(\nu_1+\nu_2-m-1)/2} \det(R)^{-\nu_1/2} \det(I_m - R)^{-\nu_2/2} \\
&\quad \times \Phi\left(\alpha; \beta; -\frac{1}{2}W^{-1/2}\Psi W^{-1/2}R^{-1}(I_m - R)^{-1}\right),
\end{aligned}$$

where $W > 0$ and $0 < R < I_m$. Now, integrating R by using (38), the marginal p.d.f. of W is obtained. □

Corollary 6.1.1. *Suppose that the joint density of the random matrices W_1 and W_2 is given by (56). Then, the p.d.f. of $S = (W_1 + W_2)^{-1}$ is given by*

$$\begin{aligned}
&\frac{\Gamma_m(\alpha)\Gamma_m[\beta - (\nu_1 + \nu_2 - 2m - 2)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha - (\nu_1 + \nu_2 - 2m - 2)/2]} \\
&\quad \times \frac{2^{-m(\nu_1+\nu_2-2m-2)/2} \det(\Psi)^{(\nu_1+\nu_2-2m-2)/2}}{\Gamma_m\left[(\nu_1 - m - 1)/2\right]\Gamma_m\left[(\nu_2 - m - 1)/2\right]} \det(S)^{(\nu_1+\nu_2-3m-3)/2}
\end{aligned}$$

$$\times\, B_m^{(\alpha,\beta)}\left(-\frac{\nu_1-m-1}{2},-\frac{\nu_2-m-1}{2};\frac{1}{2}\Psi^{1/2}S\Psi^{1/2}\right),\quad S>0.$$

Next, we will derive results like $E[C_{(1)}(S)]$, $E[C_{(2)}(S)]$, $E[C_{(1^2)}(S)]$, $E(\operatorname{tr} S)$, $E(\operatorname{tr} S^2)$, $E(\operatorname{tr} S)^2$, $E(S)$, $E(S^2)$ and $E[\operatorname{tr}(S)S]$.

Theorem 6.2. *Suppose that the joint density of the random matrices W_1 and W_2 is given by (56) with $\Psi = I_m$ and $S = (W_1+W_2)^{-1}$. Then*

$$\begin{aligned}E[C_{(1)}(S)] &= m(\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}\\ &\quad\times K_{(1)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right),\end{aligned}$$

$$\begin{aligned}E[C_{(2)}(S)] &= m(\nu_1+\nu_2-2m-2)(\nu_1+\nu_2-2m)\\ &\quad\times\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)(-\beta+(\nu_1+\nu_2-m+1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)(-\alpha+(\nu_1+\nu_2-m+1)/2)}\\ &\quad\times K_{(2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right),\end{aligned}$$

$$\begin{aligned}E[C_{(1^2)}(S)] &= m(\nu_1+\nu_2-2m-2)(\nu_1+\nu_2-2m-3)\\ &\quad\times\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)(-\beta+(\nu_1+\nu_2-m-2)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)(-\alpha+(\nu_1+\nu_2-m-2)/2)}\\ &\quad\times K_{(1^2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right),\end{aligned}$$

where $K_{(1)}$, $K_{(2)}$ and $K_{(1^2)}$ are given in Lemma 5.1, Lemma 5.2 and Lemma 5.3, respectively.

Proof. The expected value of $C_\kappa(S)$ is derived as

$$\begin{aligned}E[C_\kappa(S)] &= \frac{\Gamma_m(\alpha)\Gamma_m[\beta-(\nu_1+\nu_2-2m-2)/2]}{\Gamma_m(\beta)\Gamma_m[\alpha-(\nu_1+\nu_2-2m-2)/2]}\\ &\quad\times\frac{2^k}{\Gamma_m\left[(\nu_1-m-1)/2\right]\Gamma_m\left[(\nu_2-m-1)/2\right]}\end{aligned}$$

$$\times \int_{X>0} C_\kappa(X) \det(X)^{(\nu_1+\nu_2-3m-3)/2}$$
$$\times B_m^{(\alpha,\beta)}\left(-\frac{\nu_1-m-1}{2}, -\frac{\nu_2-m-1}{2}; X\right) \mathrm{d}X.$$

Now, setting $s=(\nu_1+\nu_2-2m-2)/2$, $a=-(\nu_1-m-1)/2$, $b=-(\nu_2-m-1)/2$ and using Corollary 5.1.1 for $\kappa=(1)$, Corollary 5.1.2 for $\kappa=(2)$, Corollary 5.1.3 for $\kappa=(1^2)$, we obtain the desired result. □

Theorem 6.3. *Suppose that the joint density of the random matrices W_1 and W_2 is given by (56) with $\Psi=I_m$ and $S=(W_1+W_2)^{-1}$. Then*

$$E[\mathrm{tr}(S)] = m(\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times K_{(1)}\left(\frac{\nu_2-m-1}{2}, \frac{\nu_1-m-1}{2}\right),$$

$$E[\mathrm{tr}(S^2)] = m(\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times \left[(\nu_1+\nu_2-2m)\frac{(-\beta+(\nu_1+\nu_2-m+1)/2)}{(-\alpha+(\nu_1+\nu_2-m+1)/2)}\right.$$
$$\times K_{(2)}\left(\frac{\nu_2-m-1}{2}, \frac{\nu_1-m-1}{2}\right)$$
$$-\frac{1}{2}(\nu_1+\nu_2-2m-3)\frac{(-\beta+(\nu_1+\nu_2-m-2)/2)}{(-\alpha+(\nu_1+\nu_2-m-2)/2)}$$
$$\left.\times K_{(1^2)}\left(\frac{\nu_2-m-1}{2}, \frac{\nu_1-m-1}{2}\right)\right],$$

$$E[(\mathrm{tr}\, S)^2] = m(\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times \left[(\nu_1+\nu_2-2m)\frac{(-\beta+(\nu_1+\nu_2-m+1)/2)}{(-\alpha+(\nu_1+\nu_2-m+1)/2)}\right.$$
$$\times K_{(2)}\left(\frac{\nu_2-m-1}{2}, \frac{\nu_1-m-1}{2}\right)$$
$$+(\nu_1+\nu_2-2m-3)\frac{(-\beta+(\nu_1+\nu_2-m-2)/2)}{(-\alpha+(\nu_1+\nu_2-m-2)/2)}$$

$$\times K_{(1^2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right)\Bigg],$$

where $K_{(1)}$, $K_{(2)}$ and $K_{(1^2)}$ are given in Lemma 5.1, Lemma 5.2 and Lemma 5.3, respectively.

Proof. We obtain the desired result by using $C_{(1)}(S) = \operatorname{tr}(S)$, $C_{(2)}(S) - C_{(1^2)}(S)/2 = (\operatorname{tr} S^2)$ and $C_{(2)}(S) + C_{(1^2)}(S) = (\operatorname{tr} S)^2$, and Theorem 6.2. □

Theorem 6.4. *Suppose that the joint density of the random matrices W_1 and W_2 is given by (56) with $\Psi = I_m$ and $S = (W_1 + W_2)^{-1}$. Then*

$$E(S) = (\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times K_{(1)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right) I_m,$$

$$E(S^2) = (\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times \Bigg[(\nu_1+\nu_2-2m)\frac{(-\beta+(\nu_1+\nu_2-m+1)/2)}{(-\alpha+(\nu_1+\nu_2-m+1)/2)}$$
$$\times K_{(2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right)$$
$$-\frac{1}{2}(\nu_1+\nu_2-2m-3)\frac{(-\beta+(\nu_1+\nu_2-m-2)/2)}{(-\alpha+(\nu_1+\nu_2-m-2)/2)}$$
$$\times K_{(1^2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right)\Bigg] I_m,$$

and

$$E[(\operatorname{tr} S)S] = (\nu_1+\nu_2-2m-2)\frac{(-\beta+(\nu_1+\nu_2-m-1)/2)}{(-\alpha+(\nu_1+\nu_2-m-1)/2)}$$
$$\times \Bigg[(\nu_1+\nu_2-2m)\frac{(-\beta+(\nu_1+\nu_2-m+1)/2)}{(-\alpha+(\nu_1+\nu_2-m+1)/2)}$$
$$\times K_{(2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right)$$

$$
\begin{aligned}
&+(\nu_1+\nu_2-2m-3)\frac{(-\beta+(\nu_1+\nu_2-m-2)/2)}{(-\alpha+(\nu_1+\nu_2-m-2)/2)} \\
&\times K_{(1^2)}\left(\frac{\nu_2-m-1}{2},\frac{\nu_1-m-1}{2}\right)\Big] I_m,
\end{aligned}
$$

where $K_{(1)}$, $K_{(2)}$ and $K_{(1^2)}$ are given in Lemma 5.1, Lemma 5.2 and Lemma 5.3, respectively.

Proof. Since, for any $m \times m$ orthogonal matrix H, the random matrices S and HSH' have the same distribution, we have $E(S) = c_1 I_m$, $E(S^2) = c_2 I_m$ and $E[(\operatorname{tr} S)S] = c_3 I_m$ and hence $E(\operatorname{tr} S) = c_1 m$, $E(\operatorname{tr} S^2) = c_2 m$ and $E[(\operatorname{tr} S)^2] = c_3 m$. Thus, the coefficient of m in the expressions for $E(\operatorname{tr} S)$, $E(\operatorname{tr} S^2)$ and $E[(\operatorname{tr} S)^2]$ are c_1, c_2 and c_3, respectively. Finally, using Theorem 6.3, we obtain the desired result. $\square$

7 Generalized extended matrix variate Beta distribution

Recently, Nagar, Roldán-Correa and Gupta [20] and Nagar and Roldán-Correa [21], by using the integrand of the extended matrix variate beta function, generalized the conventional matrix variate beta distribution and studied several of its properties. We define the generalized extended matrix variate beta density as

$$
\begin{aligned}
&\{B_m^{(\alpha,\beta)}(p,q;\Sigma)\}^{-1}\det(X)^{p-(m+1)/2}\det(I_m-X)^{q-(m+1)/2} \\
&\times \Phi\left(\alpha;\beta;-\Sigma X^{-1}(I_m-X)^{-1}\right),\quad 0<X<I_m,
\end{aligned}
$$

where $-\infty < p < \infty$, $-\infty < q < \infty$ and $\Sigma > 0$. If r and s are real numbers, then

$$
E[\det(X)^r\det(I_m-X)^s] = \frac{B_m^{(\alpha,\beta)}(p+r,q+s;\Sigma)}{B_m^{(\alpha,\beta)}(p,q;\Sigma)}.
$$

Specializing r and s in the above expression, we obtain

$$
E[\det(X)] = \frac{B_m^{(\alpha,\beta)}(p+1,q;\Sigma)}{B_m^{(\alpha,\beta)}(p,q;\Sigma)},\quad E[\det(X)^2] = \frac{B_m^{(\alpha,\beta)}(p+2,q;\Sigma)}{B_m^{(\alpha,\beta)}(p,q;\Sigma)},
$$

$$
E[\det(I_m-X)] = \frac{B_m^{(\alpha,\beta)}(p,q+1;\Sigma)}{B_m^{(\alpha,\beta)}(p,q;\Sigma)},
$$

$$E[\det(I_m - X)^2] = \frac{B_m^{(\alpha,\beta)}(p, q+2; \Sigma)}{B_m^{(\alpha,\beta)}(p, q; \Sigma)}.$$

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