# Nakayama Automorphism of Some Skew PBW Extensions

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**Abstract**

Let $R$ be an Artin-Schelter regular algebra and $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded quasi-commutative skew PBW extension over $R$. In this paper we describe the Nakayama automorphism of $A$ using the Nakayama automorphism of the ring of coefficients $R$. We calculate explicitly the Nakayama automorphism of some skew PBW extensions.

**Keywords:** Skew PBW extensions; Nakayama automorphism; Artin-Schelter regular algebras; Calabi-Yau algebras.

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## El automorfismo de Nakayama de algunas extensiones PBW torcidas

### Resumen

Sean $R$ un álgebra Artin-Schelter regular y $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ una extensión PBW torcida cuasi-conmutativa graduada sobre $R$. En este artículo se describe el automorfismo de Nakayama de $A$ usando el automorfismo de Nakayama del anillo de coeficientes $R$. También se calcula explícitamente el automorfismo de Nakayama de algunas extensiones PBW torcidas.

**Palabras clave:** Extensiones PBW torcidas; automorfismo de Nakayama; álgebras Artin-Schelter regulares; álgebras Calabi-Yau.

## 1 Introduction

There are two notions of Nakayama automorphisms: one for skew Calabi-Yau algebras and another for Frobenius algebras. In this paper, we focus on skew Calabi-Yau algebras, or equivalently, Artin-Schelter regular algebras in the connected graded case (Proposition 2.7). The Nakayama automorphism is one of important homological invariants for Artin-Schelter regular algebras algebras. Nakayama automorphisms have been studied by several authors. Reyes, Rogalski and Zhang in [1] proved three homological identities about the Nakayama automorphism and gave several applications. Liu, Wang and Wu in [2] proved that if $R$ is skew Calabi-Yau with Nakayama automorphism $\nu$ then the Ore extension $A = R[x; \sigma, \delta]$ has Nakayama automorphism $\nu'$ such that $\nu'|_R = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ with $u, b \in R$ and $u$ invertible, the parameters $u$ and $b$ are still unknown. Zhu, Van Oystaeyen and Zhang computed the Nakayama automorphisms of trimmed double Ore extensions and skew polynomial extension $R[x; \sigma]$, where $\sigma$ is a graded algebra automorphism of $R$ in terms of the homological determinant. Lü, Mao and Zhang in [3] and [4] used the Nakayama automorphism to study group actions and Hopf algebra actions on Artin-Schelter regular algebras of global dimension three and calculated explicitly the Nakayama automorphism of a class of connected graded Artin-Schelter regular algebras. Shen, Zhou and Lu in [5] described the Nakayama automorphisms of twisted tensor products of noetherian Artin-Schelter regular algebras. Liu and Ma in [6] gave an explicit formula to calculate the Nakayama automorphism of

any Ore extension $R[x; \sigma, \delta]$ over a polynomial algebra $R = \mathbb{K}[t_1, \ldots, t_m]$ for an arbitrary $m$. In general, Nakayama automorphisms are known to be tough to compute.

Skew PBW extensions and quasi-commutative skew PBW extensions were defined in [7] and are non-commutative rings of polynomial type defined by a ring and a set of variables with relations between them. Skew PBW extensions include rings and algebras coming from mathematical physics such PBW extensions, group rings of polycyclic-by-finite groups, Ore algebras, operator algebras, diffusion algebras, some quantum algebras, quadratic algebras in three variables, some 3-dimensional skew polynomial algebras, some quantum groups, some types of Auslander-Gorenstein rings, some Koszul algebras, some Calabi-Yau algebras, some Artin-Schelter regular algebras, some quantum universal enveloping algebras, and others. There are some special subclasses of skew PBW extensions such as bijective, quasi-commutative, of derivation type and of endomorphism type (see Definition 2.2). Some noncommutative rings are skew PBW extensions but not Ore extensions (see [8],[9] and [10]).

Several properties of skew PBW extensions have been studied in the literature (see for example [11],[12],[13],[14],[15],[16]). It is known that quasi-commutative skew PBW extensions are isomorphic to iterated Ore extensions of endomorphism type (see [8, Theorem 2.3]). In [9] it was defined graded skew PBW extensions with the aim of studying Koszul property in these extensions.

The Nakayama automorphism has not been studied explicitly for skew PBW extensions. In this paper we focus on the Nakayama automorphism of quasi-commutative skew PBW extensions (Section 3) and non quasi-commutative skew PBW extensions of the form $\sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x \rangle$ (Section 4). Since every graded quasi-commutative skew PBW extension is isomorphic to a graded iterated Ore extension of endomorphism type (see [17, Proposition 2.7]), we have that if $A$ is a graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra $R$, then $A$ is Artin-Schelter regular (see Proposition 3.1). Now, for $B$ a connected graded algebra, $B$ is Artin-Schelter regular if and only if $B$ is graded skew Calabi-Yau (see [1]), and hence the Nakayama automorphism of Artin-Schelter regular algebras exists. Therefore, if $R$ is an Artin-Schelter regular algebra with Nakayama automorphism $\nu$, then the Nakayama automorphism

$\mu$ of a graded quasi-commutative skew PBW extension $A$ exists, and we compute it using the Nakayama automorphism $\nu$ together some especial automorphisms of $R$ and $A$ (see Theorem 3.1). The main results in Section 3 are Theorem 3.1, Corollary 3.2, Proposition 3.3, Corollary 3.3 and Examples 3.1, 3.2. Since Ore extensions of bijective type are skew PBW extensions. In Section 4 we use the results of [6] to calculate explicitly the Nakayama automorphism of some skew PBW extensions (Examples 4.1, 4.2 and Proposition 4.1).

We establish the following notation: the symbol $\mathbb{N}$ is used to denote the set of natural numbers including zero. If $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ then $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The letter $\mathbb{K}$ denotes a field. Every algebra is a $\mathbb{K}$-algebra.

## 2  Preliminaries

In this section, we fix basic notations and recall definitions and properties for this paper.

**Definition 2.1.** Let $R$ and $A$ be rings. We say that $A$ is a *skew PBW extension over* $R$, if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \ldots, x_n \in A$ such that $A$ is a left free $R$-module, with basis the basic elements

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

In this case, it is said also that $A$ is a left polynomial ring over $R$ with respect to $\{x_1, \cdots, x_n\}$ and Mon(A) is the set of standard monomials of $A$. Moreover, $x_1^0 \cdots x_n^0 := 1 \in \mathrm{Mon}(A)$.

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R. \tag{1}$$

(iv) For any elements $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + R x_1 + \cdots + R x_n. \tag{2}$$

Under these conditions, we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

The notation $\sigma(R)\langle x_1, \ldots, x_n \rangle$ and the name of the skew PBW extensions are due to the next proposition.

**Proposition 2.1** ([7], Proposition 3)**.** Let $A$ be a skew PBW extension of $R$. For each $1 \leq i \leq n$, there exist an injective endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r) x_i + \delta_i(r), \qquad r \in R. \tag{3}$$

In the following definition we recall some sub-classes of skew PBW extensions. Examples of these sub-classes of algebras can be found in [15].

**Definition 2.2.** Let $A$ be a skew PBW extension of $R$, $\Sigma := \{\sigma_1, \ldots, \sigma_n\}$ and $\Delta := \{\delta_1, \ldots, \delta_n\}$, where $\sigma_i$ and $\delta_i$ $(1 \leq i \leq n)$ are as in Proposition 2.1

(a) $A$ is called *quasi-commutative*, if the conditions (iii) and (iv) in Definition 2.1 are replaced by

(iii') for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$, there exists $c_{i,r} \in R \setminus \{0\}$ such that
$$x_i r = c_{i,r} x_i; \tag{4}$$

(iv') for any $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that
$$x_j x_i = c_{i,j} x_i x_j. \tag{5}$$

(b) $A$ is called *bijective*, if $\sigma_i$ is bijective for each $\sigma_i \in \Sigma$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

(c) If $\sigma_i = \mathrm{id}_R$ for every $\sigma_i \in \Sigma$, we say that $A$ is a skew PBW extension of *derivation type*.

(d) If $\delta_i = 0$ for every $\delta_i \in \Delta$, we say that $A$ is a skew PBW extension of *endomorphism type*.

The next proposition was proved in [9].

**Proposition 2.2.** Let $R = \bigoplus_{m \geq 0} R_m$ be a $\mathbb{N}$-graded algebra and let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a bijective skew PBW extension of $R$ satisfying the following two conditions:

(i) $\sigma_i$ is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded $\sigma_i$-derivation for all $1 \leq i \leq n$, where $\sigma_i$ and $\delta_i$ are as in Proposition 2.1.

(ii) $x_j x_i - c_{i,j} x_i x_j \in R_2 + R_1 x_1 + \cdots + R_1 x_n$, as in (2) and $c_{i,j} \in R_0$.

For $p \geq 0$, let $A_p$ the $\mathbb{K}$-space generated by the set

$$\left\{ r_t x^\alpha \mid t + |\alpha| = p, \ r_t \in R_t \text{ and } x^\alpha \in \mathrm{Mon}(A) \right\}.$$

Then $A$ is a $\mathbb{N}$-graded algebra with graduation

$$A = \bigoplus_{p \geq 0} A_p. \tag{6}$$

**Definition 2.3** ([9], Definition 2.6)**.** Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a bijective skew PBW extension of a $\mathbb{N}$-graded algebra $R = \bigoplus_{m \geq 0} R_m$. We say that $A$ is a *graded skew PBW extension* if $A$ satisfies the conditions (i) and (ii) in Proposition 2.2.

Note that the family of graded iterated Ore extensions is strictly contained in the family of graded skew PBW extensions (see [9, Remark 2.11]). Examples of graded skew PBW extensions can be found in [9] and [10].

**Proposition 2.3.** Quasi-commutative skew PBW extensions with the trivial graduation of $R$ are graded skew PBW extensions. If we assume that $R$ has a different graduation to the trivial graduation, then $A$ is graded skew PBW extension if and only if $\sigma_i$ is graded and $c_{i,j} \in R_0$, for $1 \leq i, j \leq n$.

*Proof.* Let $R = R_0$ and $r \in R = R_0$. From (4) we have that $x_i r = c_{i,r} x_i = \sigma_i(r) x_i$. So, $\sigma_i(r) = c_{i,r} \in R = R_0$ and $\delta_i = 0$, for $1 \leq i \leq n$. Therefore $\sigma_i$ is a graded ring homomorphism and $\delta_i : R(-1) \to R$ is a graded $\sigma_i$-derivation for all $1 \leq i \leq n$. From (5) we have that $x_j x_i - c_{i,j} x_i x_j = 0 \in R_2 + R_1 x_1 + \cdots + R_1 x_n$ and $c_{i,j} \in R = R_0$. If $R$ has a nontrivial graduation, then the result is obtained from the relations (4), (5) and Definition 2.3. $\square$

Let $B = \bigoplus_{p \geq 0} B_p$ be an $\mathbb{N}$-graded algebra. $B$ is *connected* if $B_0 = \mathbb{K}$. In [18, Definition 1.4] finitely graded algebra was presented. $B$ is *finitely graded* if the following conditions hold:

(i) $B$ is $\mathbb{N}$-graded (positively graded): $B = \bigoplus_{j \geq 0} B_j$.

(ii) $B$ is connected.

(iii) $B$ is *finitely generated* as algebra, i.e., there is a finite set of elements $x_1, \ldots, x_n \in B$ such that the set $\{x_{i_1} x_{i_2} \cdots x_{i_m} \mid 1 \leq i_j \leq n, m \geq 1\} \cup \{1\}$ spans $B$ as a vector space.

Note that a finitely graded algebra $B$ is finitely generated as $K$-algebra if and only if $B = K\langle x_1, \ldots, x_m \rangle / I$, where $I$ is a proper homogeneous two-sided ideal of $K\langle x_1, \ldots, x_m \rangle$. A finitely graded algebra $B$ is *finitely presented* if the two-sided ideal $I$ is finitely generated.

The following properties of graded skew PBW extensions were proved in [9].

**Remark 2.1** ([9], Remark 2.10). Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded skew $PBW$ extension. Then we have the following properties:

(i) $A$ is a $\mathbb{N}$-graded algebra and $A_0 = R_0$.

(ii) $R$ is connected if and only if $A$ is connected.

(iii) If $R$ is finitely generated then $A$ is finitely generated.

(iv) For (i), (ii) and (iii) above, we have that if $R$ is a finitely graded algebra then $A$ is a finitely graded algebra.

(v) If $R$ is locally finite, then $A$ as $\mathbb{K}$-algebra is a locally finite.

(vi) $A$ as $R$-module is locally finite.

(vii) If $R$ is finitely presented then $A$ is finitely presented.

The following proposition establishes the relation between graded skew PBW extensions and graded iterated Ore extensions.

**Proposition 2.4** ([17], Proposition 2.7). Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded skew PBW extension. If $A$ is quasi-commutative, then $A$ is isomorphic to a graded iterated Ore extension of endomorphism type $R[z_1; \theta_1] \cdots [z_n; \theta_n]$, where $\theta_i$ is bijective, for each $i$; $\theta_1 = \sigma_1$;

$$\theta_j : R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}] \to R[z_1; \theta_1] \cdots [z_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(z_i) = c_{i,j} z_i$ ($c_{i,j} \in R_0$ as in (2)), $1 \leq i < j \leq n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$.

If $M$ and $N$ are graded $B$-modules, we use $\underline{\mathrm{Hom}}_B^d(M, N)$ to denote the set of all $B$-module homomorphisms $h : M \to N$ such that $h(M_i) \subseteq N_{i+d}$. We set $\underline{\mathrm{Hom}}_B(M, N) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_B^d(M, N)$, and we denote the corresponding derived functors by $\underline{\mathrm{Ext}}_B^i(M, N)$.

**Definition 2.4.** Let $B = \mathbb{K} \oplus B_1 \oplus B_2 \oplus \cdots$ be a finitely presented graded algebra over $\mathbb{K}$. The algebra $B$ will be called *Artin-Schelter regular*, if $B$ has the following properties:

(i) $B$ has finite global dimension $d$: every graded $B-$module has projective dimension $\leq d$.

(ii) $B$ has finite Gelfand-Kirillov dimension.

(iii) $B$ is *Gorenstein*, meaning that $\underline{\mathrm{Ext}}_B^i(\mathbb{K}, B) = 0$ if $i \neq d$, and $\underline{\mathrm{Ext}}_B^d(\mathbb{K}, B) \cong \mathbb{K}(l)$ for some $l \in \mathbb{Z}$.

The enveloping algebra of a ring $B$ is defined as $B^e := B \otimes B^{op}$. We characterize the enveloping algebra of a skew PBW extension in [12]. If $M$ is an $B$-bimodule, then $M$ is an $B^e$ module with the action given by $(a \otimes b) \cdot m = amb$, for all $m \in M$, $a, b \in B$. Given automorphisms $\nu, \tau \in \mathrm{Aut}(B)$, we can define the twisted $B^e$-module ${}^\nu M^\tau$ with the rule $(a \otimes b) \cdot m = \nu(a) m \tau(b)$, for all $m \in M$, $a, b \in B$. When one or the other of $\nu$, $\tau$ is the identity map, we shall simply omit it, writing for example $M^\nu$ for ${}^1 M^\nu$.

**Proposition 2.5** ([19], Lemma 2.1). Let $\nu$, $\sigma$ and $\phi$ be automorphisms of $B$. Then

(i) The map $^{\nu}B^{\sigma} \to {}^{\phi\nu}B^{\phi\sigma}$, $a \mapsto \phi(a)$ is an isomorphism of $B^e$-modules. In particular,

$$^{\nu}B^{\sigma} \cong B^{\nu^{-1}\sigma} \cong {}^{\sigma^{-1}\nu}B \text{ and } B^{\sigma} \cong {}^{\sigma^{-1}}B.$$

(ii) $B \cong B^{\sigma}$ as $B^e$-modules if and only if $\sigma$ is an inner automorphism.

An algebra $B$ is said to be *homologically smooth*, if as a $B^e$-module, $B$ has a finitely generated projective resolution of finite length.

**Definition 2.5.** An algebra $B$ is called *skew Calabi-Yau* of dimension $d$ if

(i) $B$ is homologically smooth.

(ii) There exists an algebra automorphism $\nu$ of $B$ such that

$$\mathrm{Ext}^i_{B^e}(B, B^e) \cong \left\{ \begin{array}{ll} 0, & i \neq d; \\ B^{\nu}, & i = d. \end{array} \right.$$

as $B^e$-modules. If $\nu$ is the identity, then $B$ is said to be *Calabi-Yau*.

A graded algebra $B$ is called *graded skew Calabi-Yau* of dimension $d$, if

(i) $B$ is homologically smooth.

(ii) There exists a graded automorphism $\nu$ of $B$ such that

$$\underline{\mathrm{Ext}}^i_{B^e}(B, B^e) \cong \left\{ \begin{array}{ll} 0, & i \neq d; \\ B^{\nu}(l), & i = d. \end{array} \right.$$

as $B^e$-graded modules, for some integer $l$. If $\nu$ is the identity, then $B$ is said to be *graded Calabi-Yau*.

The automorphism $\nu$ is called the *Nakayama* automorphism of $B$. As a consequence of Proposition 2.5, we have that a skew Calabi-Yau algebra is Calabi-Yau if and only if its Nakayama automorphism is inner. If $B$ is a Calabi-Yau algebra of dimension $d$, then the Hochschild dimension of $B$ (that is, the projective dimension of $A$ as an $A$-bimodule) is $d$ (see [20, Proposition 2.2]).

**Proposition 2.6.** Let $B$ be a skew Calabi-Yau algebra with Nakayama automorphism $\nu$. Then $\nu$ is unique up to an inner automorphism, i.e, the Nakayama automorphism is determined up to multiplication by an inner automorphism of $B$.

*Proof.* Let $B$ be a skew Calabi-Yau algebra with Nakayama automorphism $\nu$ and let $\mu$ be another Nakayama automorphism, i.e., $\operatorname{Ext}^d_{B^e}(B, B^e) \cong B^\mu$, then $\operatorname{Ext}^d_{B^e}(B, B^e) \cong B^\nu \cong B^\mu$ as $B^e$ -modules. By Proposition 2.5-(i), $B \cong B^{\nu^{-1}\mu}$; by Proposition 2.5-(ii), $\nu^{-1}\mu$ is an inner automorphism of $B$. Let $\nu^{-1}\mu = \sigma$ where $\sigma$ is an inner automorphism of $B$, so $\mu = \nu\sigma$ for some inner automorphism $\sigma$ of $B$. $\qquad\square$

Reyes, Rogalski and Zhang in [1], proved that Artin-Schelter regular algebras and graded skew Calabi-Yau algebras coincide for the connected case.

**Proposition 2.7** ([1], Lemma 1.2)**.** Let $B$ be a connected graded algebra. Then $B$ is graded skew Calabi-Yau if and only if it is Artin-Schelter regular.

## 3 Nakayama automorphism of quasi-commutative skew PBW extensions

Since graded quasi-commutative skew PBW extensions are isomorphic to graded iterated Ore extensions of endomorphism type, it follows that graded quasi-commutative skew PBW extensions with coefficients in Artin-Schelter regular algebras are skew Calabi-Yau, and the Nakayama automorphism exists for these extensions. With this in mind, in this section we give a description of Nakayama automorphism for these non-commutative algebras using the Nakayama automorphism of the ring of the coefficients.

The following properties of graded quasi-commutative skew PBW extensions were proved in [17, Theorem 3.6 and Theorem 4.5].

**Proposition 3.1.** Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded quasi-commutative skew PBW extension of $R$.

(i) If $R$ is Artin-Schelter regular, then $A$ is Artin-Schelter regular.

(ii) If $R$ is a finitely presented graded skew Calabi-Yau algebra of global dimension $d$, then $A$ is graded skew Calabi-Yau of global dimension $d + n$.

As a consequence of Proposition 2.7 and Proposition 3.1 we have the following property.

**Corollary 3.1.** Let $R$ be an Artin-Schelter regular algebra of global dimension $d$. Then every graded quasi-commutative skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is graded skew Calabi-Yau of global dimension $d+n$.

Liu, Wang and Wu in [2, Theorem 3.3] proved that if $R$ is a skew Calabi-Yau algebra with Nakayama automorphism $\nu$ then the Ore extension $A = R[x; \sigma, \delta]$ has Nakayama automorphism $\nu'$ such that $\nu'|_R = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ with $u, b \in R$ and $u$ invertible. If $\delta = 0$ then they deduced the following result.

**Proposition 3.2** ([2], Remark 3.4)**.** With the above notation $\nu'(x) = ux$ if $\delta = 0$.

In the next theorem we show a way to calculate the Nakayama automorphism for a graded quasi-commutative skew PBW extension $A$ of an Artin-Schelter regular algebra $R$.

**Theorem 3.1.** Let $R$ be an Artin-Schelter regular algebra with Nakayama automorphism $\nu$. Then the Nakayama automorphism $\mu$ of a graded quasi-commutative skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is given by

$$\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1}\nu(r), \text{ for } r \in R, \text{ and}$$

$$\mu(x_i) = u_i \prod_{j=i}^{n} c_{i,j}^{-1} x_i, \text{ for each } 1 \leq i \leq n,$$

where $\sigma_i$ is as in Proposition 2.1, $u_i, c_{i,j} \in \mathbb{K} \setminus \{0\}$, and the elements $c_{i,j}$ are as in Definition 2.1.

*Proof.* Note that $A$ is graded skew Calabi-Yau (see Corollary 3.1) and therefore the Nakayama automorphism of $A$ exists. By Proposition 2.4 we have

that $A$ is isomorphic to a graded iterated Ore extension $R[x_1; \theta_1] \cdots [x_n; \theta_n]$, where $\theta_i$ is bijective; $\theta_1 = \sigma_1$;

$$\theta_j : R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}] \to R[x_1; \theta_1] \cdots [x_{j-1}; \theta_{j-1}]$$

is such that $\theta_j(x_i) = c_{i,j} x_i$ ($c_{i,j} \in \mathbb{K}$ as in Definition 2.1), $1 \le i < j \le n$ and $\theta_i(r) = \sigma_i(r)$, for $r \in R$. Note that

$$\theta_j^{-1}(x_i) = c_{i,j}^{-1} x_i. \tag{7}$$

Now, since $R$ is connected then by Remark 2.1, $A$ is connected. So, the multiplicative group of $R$ and also the multiplicative group of $A$ is $\mathbb{K} \setminus \{0\}$, therefore the identity map is the only inner automorphism of $A$. Let $\mu_i$ the Nakayama automorphism of $R[x_1; \theta_1] \cdots [x_i; \theta_i]$.

By Proposition 3.2 we have that the Nakayama automorphism $\mu_1$ of $R[x_1; \theta_1]$ is given by $\mu_1(r) = \sigma_1^{-1} \nu(r)$ for $r \in R$, and $\mu_1(x_1) = u_1 x_1$ with $u_1 \in \mathbb{K} \setminus \{0\}$; the Nakayama automorphism $\mu_2$ of $R[x_1; \theta_1][x_2; \theta_2]$ is given by $\mu_2(r) = \sigma_2^{-1} \mu_1(r) = \sigma_2^{-1} \sigma_1^{-1} \nu(r)$, for $r \in R$; $\mu_2(x_1) = \theta_2^{-1} \mu_1(x_1) = \theta_2^{-1}(u_1 x_1) = u_1 \theta_2^{-1}(x_1) = u_1 c_{1,2}^{-1} x_1$ and $\mu_2(x_2) = u_2 x_2$, for $u_2 \in \mathbb{K} \setminus \{0\}$; the Nakayama automorphism $\mu_3$ of $R[x_1; \theta_1][x_2; \theta_2][x_3; \theta_3]$ is given by $\mu_3(r) = \sigma_3^{-1} \mu_2(r) = \sigma_3^{-1} \sigma_2^{-1} \sigma_1^{-1} \nu(r)$, for $r \in R$; $\mu_3(x_1) = \theta_3^{-1} \mu_2(x_1) = \theta_3^{-1}(u_1 c_{1,2}^{-1} x_1) = u_1 c_{1,2}^{-1} \theta_3^{-1}(x_1) = u_1 c_{1,2}^{-1} c_{1,3}^{-1} x_1$; $\mu_3(x_2) = \theta_3^{-1} \mu_2(x_2) = \theta_3^{-1}(u_2 x_2) = u_2 c_{2,3}^{-1} x_2$ and $\mu_3(x_3) = u_3 x_3$, for $u_3 \in \mathbb{K} \setminus \{0\}$.

Continuing with the procedure we have that the Nakayama automorphism of $A$ is given by

$$\mu(r) = \mu_n(r) = \sigma_n^{-1} \cdots \sigma_2^{-1} \sigma_1^{-1} \nu(r) = (\sigma_1 \cdots \sigma_n)^{-1} \nu(r),$$

for $r \in R$, and

$$\mu(x_1) = \mu_n(x_1) = u_1 c_{1,2}^{-1} c_{1,3}^{-1} \cdots c_{1,n}^{-1} x_1;$$
$$\mu(x_2) = \mu_n(x_2) = u_2 c_{2,3}^{-1} \cdots c_{2,n}^{-1} x_2; \cdots .$$

In general, for $1 \le i \le n$, we have that

$$\mu(x_i) = u_i c_{i,i+1}^{-1} c_{i,i+2}^{-1} \cdots c_{i,n}^{-1} x_i, \quad \text{for } u_i \in \mathbb{K} \setminus \{0\}.$$

Note that $c_{i,i} = 1$. $\qquad \square$

From Theorem 3.1 the following corollary is immediately obtained.

**Corollary 3.2.** Let $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ be a graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra $R$ with Nakayama automorphism $\nu$. Then $A$ is graded Calabi-Yau if and only if $\sigma_1 \cdots \sigma_n = \nu$ and $u_i = \prod_{j=i}^{n} c_{i,j}$, for all $1 \leq i \leq n$, where $u_i$ is as in Theorem 3.1.

Let $\mathbb{K}[t_1, \ldots, t_m]$ be the polynomial algebra and $\sigma : \mathbb{K}[t_1, \ldots, t_m] \to \mathbb{K}[t_1, \ldots, t_m]$ an algebra automorphism. We define

$$
M := \begin{pmatrix}
\frac{\partial \sigma(t_1)}{\partial t_1} & \frac{\partial \sigma(t_1)}{\partial t_2} & \cdots & \frac{\partial \sigma(t_1)}{\partial t_m} \\
\frac{\partial \sigma(t_2)}{\partial t_1} & \frac{\partial \sigma(t_2)}{\partial t_2} & \cdots & \frac{\partial \sigma(t_2)}{\partial t_m} \\
\vdots & \vdots & \vdots & \vdots \\
\frac{\partial \sigma(t_m)}{\partial t_1} & \frac{\partial \sigma(t_n)}{\partial t_2} & \cdots & \frac{\partial \sigma(t_m)}{\partial t_m}
\end{pmatrix}.
\tag{8}
$$

**Remark 3.1.** Liu and Ma proved in [6] that if $A = \mathbb{K}[t_1, \ldots, t_m][x; \sigma]$ is an Ore extension then the Nakayama automorphism $\nu$ of $A$ is given by $\nu(t_i) = t_i$ and $\nu(x) = \det(M)x$, where $\det(M)$ is the usual determinant of the matrix $M$ in (8). Note that $A = \mathbb{K}[t_1, \ldots, t_m][x; \sigma]$ is a quasi-commutative skew PBW extension and if $\sigma$ is a graded automorphism then $A$ is a graded quasi-commutative skew PBW extension.

Let $h \in \mathbb{K} \setminus \{0\}$. The *algebra of shift operators* is defined by $S_h := \mathbb{K}[t][x_h; \sigma_h]$, where $\sigma_h(p(t)) := p(t - h)$. Notice that $x_h t = (t - h)x_h$ and for $p(t) \in \mathbb{K}[t]$ we have $x_h^i p(t) = p(t - ih)x_h^i$. Thus, $S_h \cong \sigma(\mathbb{K}[t])\langle x_h \rangle$ is a quasi-commutative skew $PBW$ extension of $\mathbb{K}[t]$.

**Proposition 3.3.** The algebra of shift operators is a Calabi-Yau algebra.

*Proof.* Since $x_h t = (t - h)x_h$, then $\sigma_h(t) := t - h$, then $\det(M) = \frac{\partial t - h}{\partial t} = 1$. By Remark 3.1 we have that the Nakayama automorphism $\nu$ of $S_h$ is given by $\nu(t) = t$ and $\nu(x_h) = \det(M)x = 1$, i.e., the Nakayama automorphism of $S_h$ is the identity map. Therefore $S_h$ is Calabi-Yau. $\square$

**Corollary 3.3.** Let $A = \mathbb{K}[t_1, \ldots, t_m][x; \sigma]$ with $\sigma$ be a graded algebra automorphism. Then $u = \det(M)$, where $u$ is as in Theorem 3.1 and $M$ is as in (8).

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*Proof.* Note that $A = \mathbb{K}[t_1, \ldots, t_m][x; \sigma]$ is a quasi-commutative skew PBW extension of the graded Calabi-Yau algebra $\mathbb{K}[t_1, \ldots, t_m]$. Since $\sigma$ is graded then $A$ is a graded quasi-commutative skew PBW extension. By Theorem 3.1 we have that the Nakayama automorphism $\mu$ of $A$ is given by

$$\mu(t_i) = t_i, \text{ for } 1 \leq i \leq m, \text{ and}$$
$$\mu(x) = ux, \text{ for some } u \in \mathbb{K} \setminus \{0\}.$$

By Remark 3.1 we have that the Nakayama automorphism of $A$ is given by

$$\nu(t_i) = t_i \text{ and}$$
$$\nu(x) = \det(M)x.$$

Since $\mathbb{K}[t_1, \ldots, t_m]$ is connected, then by Remark 2.1 $A$ is connected, so the multiplicative group of $A$ is $\mathbb{K} \setminus \{0\}$. Therefore the identity map is the only inner automorphism of $A$. As by Proposition 2.6 we have that the Nakayama automorphism is unique up to inner automorphisms, then $\nu$ must be equal to $\mu$. So $u = \det(M)$. $\qquad\square$

**Example 3.1.** Let $R$ be an Artin-Schelter regular algebra of global dimension $d$ with Nakayama automorphism $\nu$. Let

$$A = R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n]$$

be an iterated skew polynomial ring (see [21, Page 23]), with $\sigma_i$ graded. $A$ is a skew PBW extension of $R$ with relations $x_i r = \sigma_i(r)x_i$ and $x_j x_i = x_i x_j$, for $r \in R$ and $1 \leq i, j \leq n$. As $R$ is graded and $c_{i,j} = 1 \in R_0$, then by Proposition 2.3 we have that $A$ is a graded quasi-commutative skew PBW extension of $R$. Therefore, $A$ is a graded skew Calabi-Yau algebra (Corollary 3.1). By Proposition 2.4, $R[x_1, \ldots, x_n; \sigma_1, \ldots, \sigma_n] \cong R[x_1; \theta_1] \cdots [x_n; \theta_n]$, where $\theta_j(r) = \sigma_j(r)$ and $\theta_j(x_i) = x_i$ for $i < j$. Applying Theorem 3.1 we have that the Nakayama automorphism $\mu$ of $A$ is given by $\mu(r) = (\sigma_1 \cdots \sigma_n)^{-1}\nu(r)$, if $r \in R$ and $\mu(x_i) = u_i \prod_{j=i}^n c_{i,j}^{-1} x_i = u_i x_i$, $u_i \in \mathbb{K} \setminus \{0\}$, $1 \leq i \leq n$.

**Example 3.2.** For a fixed $q \in \mathbb{K} - \{0\}$, the *algebra of linear partial q-dilation operators* $H$, with polynomial coefficients, is the free algebra $\mathbb{K}\langle t_1, \ldots, t_n, H_1^{(q)}, \ldots, H_m^{(q)} \rangle$, $n \geq m$, subject to the relations:

$$t_j t_i = t_i t_j, \quad 1 \leq i < j \leq n;$$

$$H_i^{(q)} t_i = q t_i H_i^{(q)}, \quad 1 \le i \le m;$$

$$H_j^{(q)} t_i = t_i H_j^{(q)}, \quad i \ne j;$$

$$H_j^{(q)} H_i^{(q)} = H_i^{(q)} H_j^{(q)}, \quad 1 \le i < j \le m.$$

The algebra $H$ is a graded quasi-commutative skew PBW extension of $\mathbb{K}[t_1, \ldots, t_n]$, where $\mathbb{K}[t_1, \ldots, t_n]$ is endowed with usual graduation. According to Proposition 2.1, we have that

$$\sigma_j(t_i) = t_i \text{ for } i \ne j, \quad 1 \le i \le n, \quad 1 \le j \le m;$$

$$\sigma_j(t_i) = q t_i \text{ for } i = j, \quad 1 \le i, j \le m;$$

$$\delta_j = 0, \text{ for } 1 \le j \le m.$$

By Proposition 2.4, $H$ is isomorphic to a graded iterated Ore extension of endomorphism type

$$\mathbb{K}[t_1, \ldots, t_n][H_1^{(q)}; \theta_1] \cdots [H_{m-1}^{(q)}; \theta_{m-1}][H_m^{(q)}; \theta_m],$$

with

$$\theta_j(t_i) = t_i \text{ for } i \ne j, \quad 1 \le i \le n, \quad 1 \le j \le m;$$

$$\theta_j(t_i) = q t_i \text{ for } i = j, \quad 1 \le i, j \le m;$$

$$\theta_j(H_i^{(q)}) = H_i^{(q)} \text{ for } 1 \le i, j \le m.$$

Since $\mathbb{K}[t_1, \ldots, t_n]$ is a graded Calabi-Yau algebra, then its Nakayama automorphism $\nu$ is the identity map. Applying Theorem 3.1, we have that the Nakayama automorphism $\mu$ of $H$ is given by

$$\mu(t_i) = \sigma_m^{-1} \cdots \sigma_1^{-1}(t_i) = \begin{cases} \sigma_i^{-1}(t_i) = q^{-1} t_i, & 1 \le i \le m; \\ t_i, & m < i \le n, \end{cases}$$

$$\mu(H_j^{(q)}) = u_j H_j^{(q)} \text{ for } u_j \in \mathbb{K} \setminus \{0\}, \ 1 \le j \le m.$$

If $m = 1$, then $H = \mathbb{K}[t_1, \ldots, t_n][H_1^{(q)}; \theta_1]$ with $\theta_1(t_i) = \begin{cases} q t_1, & i = 1; \\ t_i, & 1 < i \le n. \end{cases}$

Therefore $M = \begin{pmatrix} q & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$

So, $\det(M) = q$. By Corollary 3.3 we have that the Nakayama automorphism $\mu$ of $H$ is given by

$$\mu(t_i) = \begin{cases} q^{-1}t_i, & i = 1 \\ t_i, & 1 < i \le n, \end{cases}$$
$$\mu(H_1^{(q)}) = qH_1^{(q)}.$$

Therefore, the Nakayama automorphism of $H$ is completely determined by the parameter $q$.

## 4 Nakayama automorphism of some specific skew PBW extensions

Liu and Ma in [6] computed the Nakayama automorphism for Ore extensions over a polynomial algebra in $n$ variables. Since Ore extensions of bijective type are skew PBW extensions, in this section we use these results to calculate explicitly the Nakayama automorphism of some skew PBW extensions of the form $A = \sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x \rangle$, which are not quasi-commutative.

Next we present [6, Theorem 4.2], but using the notation of skew PBW extensions.

**Theorem 4.1.** Let $A = \sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x \rangle$ be a skew PBW extension of derivation type. The Nakayama automorphism $\nu$ of $A$ is then given by $\nu(r) = r$ for all $r \in R$ and $\nu(x) = x + \sum_{i=1}^{n} \frac{\partial \delta(t_i)}{\partial t_i}$, where $\delta$ is as in Proposition 2.1.

Next we calculate the Nakayama automorphism of a skew PBW extension of derivation type using Theorem 4.1. This automorphism had already been calculated by other authors using other techniques (see for example [2] or [22]).

**Example 4.1.** Let $A = \mathbb{K}\langle x, y \rangle / \langle yx - xy - x^2 \rangle$ be the Jordan plane. Note that $A = \mathbb{K}[x][y; \delta]$ is a graded skew PBW extension of $\mathbb{K}[x]$ of derivation type, with $\delta(x) = x^2$. By Theorem 4.1 we have that the Nakayama

automorphism of $A$ is given by

$$\nu(x) = x \quad\text{and}\quad \nu(y) = y + \frac{\partial\delta(x)}{\partial x} = y + \frac{\partial x^2}{\partial x} = y + 2x.$$

If the Nakayama automorphism of a skew Calabi-Yau algebra $A$ is the identity then $A$ is Calabi-Yau. Thus, from Theorem 4.1 the following are obtained immediately.

**Corollary 4.1.** Let $A = \sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x \rangle$ be a skew PBW extension. Then $A$ is Calabi-Yau if and only if $\sigma = id$ and $\sum_{i=1}^{n} \frac{\partial\delta(t_i)}{\partial t_i} = 0$.

**Example 4.2.** The first Weyl algebra $A_1(\mathbb{K}) = \mathbb{K}[t][x; \partial/\partial t]$ is a skew PBW extension of derivation type which is not graded. Note that $xt = tx + 1$, then $\delta(t) = 1$ and $\frac{\partial\delta(t)}{\partial t} = \frac{\partial 1}{\partial t} = 0$. By Corollary 4.1 we have that $A_1(\mathbb{K})$ is a Calabi-Yau algebra, which was already known but using other techniques (see for example [23]).

Liu and Ma in [6, Lemma 4.4] proved that for the Ore extension $\mathbb{K}[t_1, \ldots, t_m][x; \sigma, \delta]$ with $\sigma \neq id$, there is a unique $k$ in the quotient field $\mathbb{K}[t_1, \ldots, t_m][x; \sigma, \delta]_q$ of $\mathbb{K}[t_1, \ldots, t_m][x; \sigma, \delta]$ such that

$$\delta(h) = k(\sigma(h) - h) \quad \text{for all } h \in R. \tag{9}$$

Next we present [6, Theorem 4.5], but using the notation of skew PBW extensions.

**Theorem 4.2.** Let $A = \sigma(\mathbb{K}[t_1, \ldots, t_m])\langle x \rangle$ be a graded skew PBW extension with $\sigma \neq id$. Then the Nakayama automorphism $\nu$ of $A$ is given by

$$\nu(r) = \begin{cases} \sigma^{-1}(r), & r \in R; \\ \det(M)x + \det(M)k - \sigma_q^{-1}(k), & r = x. \end{cases}$$

where $M$ is as in (8), $k$ is as in (9) and $\sigma_q$ is the extension of $\sigma$ to $\mathbb{K}[t_1, \ldots, t_m]_q$.

Lü, Mao and Zhang in [3] calculated the Nakayama automorphism for the following classes of algebras:
$A(1) = \mathbb{K}\langle t_1, t_2, t_3\rangle / \langle t_2 t_1 - p_{12} t_1 t_2, t_3 t_1 - p_{13} t_1 t_3, t_3 t_2 - p_{23} t_2 t_3 \rangle,$

$A(2) = \mathbb{K}\langle t_1, t_2, t_3 \rangle / \langle t_1 t_2 - t_2 t_1, \ t_1 t_3 - t_3 t_1, \ t_3 t_2 - p t_2 t_3 - t_1^2 \rangle$,

$A(3) = \mathbb{K}\langle t_1, t_2, t_3 \rangle / \langle (t_2 + t_1) t_1 - t_1 t_2, \ t_3 t_1 - q t_1 t_3, \ t_3 t_2 - q(t_2 + t_1) t_3 \rangle$,

$A(4) = \mathbb{K}\langle t_1, t_2, t_3 \rangle / \langle (t_2 + t_1) t_1 - t_1 t_2, \ t_3 t_1 - p t_1 t_3, \ t_3 t_2 - p t_2 t_3 \rangle$,

$A(5) = \mathbb{K}\langle t_1, t_2, t_3 \rangle / \langle (t_2 + t_1) t_1 - t_1 t_2, \ (t_3 + t_2 + t_1) t_1 - t_1 t_3, \ (t_3 + t_2 + t_1) t_2 - (t_2 + t_1) t_3 \rangle$,

where $\mathbb{K}$ is an algebraically closed field of characteristic zero and $p_{12}, p_{13}, p_{23}, p, q \in \mathbb{K} \setminus \{0\}$. Notice that these algebras are graded skew PBW extensions. In the following proposition we calculate the Nakayama automorphism of $A(2)$ using Theorem 4.2.

**Proposition 4.1.** The Nakayama automorphism $\nu$ of $A(2)$ is given by $\nu(t_1) = t_1$, $\nu(t_2) = p^{-1} t_2$, $\nu(t_3) = p t_3$.

*Proof.* Since $t_2 t_1 = t_1 t_2$; $\quad t_3 t_1 = t_1 t_3$ and $\quad t_3 t_2 = p t_2 t_3 - t_1^2$, then $A(2)$ is a graded skew PBW extension of $\mathbb{K}[t_1, t_2]$, i.e., $A(2) = \sigma(\mathbb{K}[t_1, t_2])\langle t_3 \rangle$. Note that $\sigma(t_1) = t_1$, $\sigma(t_2) = p t_2$, $\delta(t_1) = 0$ and $\delta(t_2) = -t_1^2$, where $\sigma$ and $\delta$ are as in Proposition 2.1. Thus,

$$M = \begin{pmatrix} \frac{\partial t_1}{\partial t_1} & \frac{\partial t_1}{\partial t_2} \\ \frac{\partial p t_2}{\partial t_1} & \frac{\partial p t_2}{\partial t_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \quad \text{and} \quad \det(M) = p.$$

Note that $k = -\frac{t_1^2}{p t_2 - t_2}$, $\sigma_q(k) = -\frac{t_1^2}{p(p t_2 - t_2)}$ and $\sigma_q^{-1}(k) = -p \frac{t_1^2}{(p t_2 - t_2)}$. Then by Theorem 4.2 $\nu(t_1) = \sigma^{-1}(t_1) = t_1$, $\nu(t_2) = \sigma^{-1}(t_2) = p^{-1} t_2$ and $\nu(t_3) = \det(M) t_3 + \det(M) k - \sigma_q^{-1}(k) = p t_3 - p \frac{t_1^2}{p t_2 - t_2} + p \frac{t_1^2}{(p t_2 - t_2)} = p t_3$. $\qquad \square$

The Nakayama automorphism of Proposition 4.1 coincides with the Nakayama automorphism in [3, Equation (E1.5.2)].

# 5 Conclusions

There are some special subclasses of skew PBW extensions such as bijective, quasi-commutative, of derivation type and of endomorphism type (see Definition 2.2). Some noncommutative rings are skew PBW extensions but not Ore extensions (see [8],[9] and [10]). We illustrate each one of the properties studied here with some of these examples of skew PBW extensions.

From Proposition 3.1, if $A$ is a graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra $R$, then $A$ is Artin-Schelter regular. Now, for connected graded algebras, an algebra is Artin-Schelter regular if and only if this is graded skew Calabi-Yau (see 2.7). Therefore, if $R$ is an Artin-Schelter regular algebra with Nakayama automorphism $\nu$, then the Nakayama automorphism $\mu$ of a graded quasi-commutative skew PBW extension $A$ exists, and we compute it using the Nakayama automorphism $\nu$ together some especial automorphisms of $R$ and $A$ (see Theorem 3.1). As a consequence of Theorem 3.1, we have that if $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$ is a graded quasi-commutative skew PBW extension of an Artin-Schelter regular algebra $R$ with Nakayama automorphism $\nu$, then $A$ is graded Calabi-Yau if and only if $\sigma_1 \cdots \sigma_n = \nu$ and $u_i = \prod_{j=i}^{n} c_{i,j}$, for all $1 \leq i \leq n$ (see Corollary 3.2). Another consequence of Theorem 3.1 is the fact that for $A = \mathbb{K}[t_1, \ldots, t_m][x; \sigma]$ with $\sigma$ be a graded algebra automorphism, $u = \det(M)$, i.e., the element $u$ is calculable (see Corollary 3.3).

Most authors give a description of the Nakayama automorphism of some specific Artin-Schelter regular algebras in terms of some parameters that can not be calculated explicitly (see for example [2],[5] and [24]). Since Ore extensions of bijective type are skew PBW extensions, in Section 4 is calculated explicitly the Nakayama automorphism of some skew PBW extensions (Examples 4.1, 4.2 and Proposition 4.1).

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