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A Survey on Some Algebraic Characterizations of Hilbert's Nullstellensatz for Non-commutative Rings of Polynomial Type

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Abstract

In this paper we present a survey of some algebraic characterizations of Hilbert's Nullstellensatz for non-commutative rings of polynomial type. Using several results established in the literature, we obtain a version of this theorem for the skew Poincaré-Birkhoff-Witt extensions. Once this is done, we illustrate the Nullstellensatz with examples appearing in non-commutative ring theory and non-commutative algebraic geometry.

Keywords: Hilbert's Nullstellensatz; skew PBW extension; Jacobson ring; generic flatness.

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Un estudio sobre algunas caracterizaciones algebraicas del teorema de ceros de Hilbert para anillos no conmutativos de tipo polinomial

Resumen
En este artículo presentamos un estudio sobre algunas caracterizaciones algebraicas del teorema de Nullstellensatz de Hilbert para anillos no conmutativos de tipo polinomial. Utilizando varios resultados establecidos en la literatura, obtuvimos una versión de este teorema para las extensiones de Poincaré-Birkhoff-Witt. Una vez hecho esto, ilustramos el Nullstellensatz con ejemplos que aparecen en la teoría de los anillos no conmutativa y en la geometría algebraica no conmutativa.

Palabras clave: Teorema de ceros de Hilbert; extensión PBW torcida; anillo de Jacobson; plenitud genérica.

1 Introduction

Hilbert's Nullstellensatz is one of the three fundamental theorems about polynomial rings over fields. The two other theorems are Hilbert's basis theorem that asserts that polynomial rings over fields are Noetherian, and Hilbert's syzygy theorem that concerns the relations, or syzygies in Hilbert's terminology, between the generators of an ideal, or, more generally, a module. Several formulations of the Nullstellensatz have been given throughout history. The most important formulation establishes a relationship between the radical of a polynomial ideal and the ideal of a variety of a polynomial ideal over the affine space. More precisely,

Proposition 1.1 ([1]. Theorem 4.2.6. (Strong Nullstellensatz))**.** Let $\mathbb{F}$ be an algebraically closed field. If I is an ideal in $\mathbb{F}[x_1, \dots, x_n]$, then $\mathbf{I}(\mathbf{V}((I)) = \sqrt{I}$.

If we think of a version of the Hilbert's Nullstellensatz for non-commutative rings of polynomial type, and throughout we can notice that Proposition 1.1 has several problems when we want to define a notion of variety since we have to switch the indeterminates, for this reason, in this paper we adopt an algebraic point of view with the aim of establishing the theorem for non-commutative algebraic structures. Our starting point is the following key fact about algebraic extensions of fields which is considered as the Hilbert's Nullstellensatz in the commutative case. In fact Proposition 7.9 in Atiyah's book is a previous result to the weak Nullstellensatz, which is Corollary 7.10.

Proposition 1.2 ([2], Proposition 7.9. (Hilbert's Nullstellensatz))**.** Let $\Bbbk$ be a field and E a finitely generated $\Bbbk$-algebra. If E is a field then it is a finite algebraic extension of $\Bbbk$.

We can note that this version of Hilbert's Nullstellensatz does not use the notion of variety; it only includes algebraic properties.

With this result in mind, our purpose in this paper is to present several algebraic formulations to Hilbert's Nullstellensatz which have been given in the literature and cross out remarkable examples of non-commutative algebras appearing in mathematics and theoretical physics. More precisely, in Section 2 we consider all these algebraic formulations following their historical development. Finally, in Section 3, we illustrate the theorem considering the family of non-commutative rings known as skew Poincaré-Birkhoff-Witt extensions.

2 Algebraic formulations

One of the most important approaches for the non-commutative formulation of the Nullstellensatz is presented by Irving in [3]. There, we can find a relation between Hilbert's Nullstellensatz for the Ore extensions defined by Ore [4] and the generic flatness condition (see Definition 2.5). Let us see the details.

Definition 2.1 ([5], page 60)**.** Let R be a ring and I an ideal of R. We say that I is a *left* (*right*) *primitive ideal*, if there exists a simple left (right) R-module M such that $\text{Ann}(M) = I$ (the annihilator of M). A *right* (*left*) *primitive ring* is any ring in which 0 is a right (left) primitive ideal, i.e., any ring which has a faithful simple right (left) module.

Definition 2.2 ([6], page 342)**.** R is a *Jacobson ring*, if every prime ideal in R is an intersection of primitive ideals.

In particular, the nilradical of any ideal in a Jacobson ring coincides with the Jacobson radical. This property is the usual definition of a Jacobson ring ([2], page 71) and it is the one used in the context of non-commutative rings. In addition, there are other characterizations of Jacobson rings. For example, a ring R is a Jacobson ring if $J(R/P) = 0$ (the Jacobson radical), for every prime ideal P of R. With this definition, we can see some examples of Jacobson rings, such as $\mathbb{Z}$ or $\Bbbk[x]$, with $\Bbbk$ a field. Of course, note that every field is a Jacobson ring.

It is well-known that for commutative rings primitive ideals coincide with maximal ideals, whence a Jacobson ring is one in which every prime ideal is an intersection of maximal ideals. The most important property of the Jacobson ring establishes a relationship between a ring and its ring of polynomials as the following proposition shows.

Proposition 2.1 ([3], Proposition 2)**.** If R is a Jacobson ring, so is the polynomial ring $R[x]$.

Proposition 2.1 can be used to deduce the Hilbert's Nullstellensatz, and it could be considered a version of the Nullstellensatz. We might think that for the non-commutative rings introduced by Ore [4], if we start from a Jacobson ring and the extension of this ring is also a Jacobson ring we could conclude the Nullstellensatz. Unfortunately, at least in the Ore extensions, this result does not hold. Several authors have shown that non-commutative extensions of Jacobson rings which are not necessarily Jacobson (e.g., [3],[7] and [8]). In [9], under suitable conditions, Nasr-Isfahani and Moussavi proved that an Ore extension of a Jacobson ring is also Jacobson. This fact tells us that to extend Jacobson's property we have to consider some other notions.

First of all, recall that a prime ideal P of a ring B (not necessarily commutative) is said to be a G-*ideal*, if the primes which properly contain P intersect in an ideal properly containing P. In this way, a commutative ring R is a G-domain if $\{0\}$ is a G-ideal (the G-ideals of a commutative ring intersect in the nilradical), and R is called a *Jacobson ring* if every G-ideal is maximal (for more details, see [3], p. 260).

Definition 2.3 ([3], page 261)**.** Let R be a commutative ring and A an R-algebra. We say that A *satisfies the* (*right*) *strong Nullstellensatz*, if for every simple right A-module M, the annihilator I intersects R in a G-ideal.

Definition 2.4 ([3], page 262)**.** Let A be an algebra over a field $\Bbbk$. It is said that A *satisfies the Nullstellensatz*, if for any simple right A-module M, the division ring $\text{End}_A(M)$ is algebraic over $\Bbbk$.

Definition 2.4 extends Proposition 1.2: Let K be a finitely-generated algebra over $\Bbbk$. Then K is a simple K-module, which equals its own endomorphism ring, and hence we have the algebraic version of the Hilbert's Nullstellensatz.

The use of the word "strong" is justified by Proposition 2.2.

Proposition 2.2 ([3], Proposition 3)**.** Let A be an algebra over a field $\Bbbk$. If $A[x]$ satisfies the strong Nullstellensatz as a $\Bbbk[x]$-algebra, then A satisfies the Nullstellensatz.

Proof. We follow the ideas presented by Irving in [3] with the aim of showing the importance of G-ideals. Let M be a simple A-module and let ϕ be an A-endomorphism of M. Note that M is an $A[x]$-module with x acting as ϕ does. By Definition 2.3, some G-ideal of $\Bbbk[x]$ annihilates M, and since $\Bbbk[x]$ is not a G-domain then every G-ideal of $\Bbbk[x]$ is maximal, whence for some non-zero polynomial $p(x) \in \Bbbk[x]$, $Mp(x) = 0$. Of course, this fact means that $p(x)$ is the zero endomorphism, that is, ϕ is algebraic over $\Bbbk$. □

We have a more general version of the Nullstellensatz, even for the non-commutative case. However, in general, it is not easy to verify. Hence we will develop several tools in order to make such verification easier. For that, we recall the notion of generic flatness.

Definition 2.5 ([3], page 263)**.** We say that an algebra A over a commutative domain R satisfies *generic flatness*, if for any finitely-generated A-module M, there exists a non-zero element $c \in R$ such that the module $M_c = M \otimes_R R_c$ is free over the localization R_c.

We identify a relationship between generic flatness and the Nullstellensatz. This interaction is presented in [10] and justified in [3].

Proposition 2.3 ([3], Proposition 4)**.** Let R be a commutative ring and A an $R-$algebra. Suppose that for each prime P of R, the R/P-algebra A/PA satisfies generic flatness. Then A satisfies the strong Nullstellensatz.

Propositions 2.2 and 2.3 imply the following corollary.

Corollary 2.1 ([3], page 266)**.** Let A be an algebra over the field $\Bbbk$ and assume that $A[x]$ satisfies generic flatness as a $\Bbbk[x]-$algebra. Then A satisfies the Nullstellensatz.

The following proposition establishes sufficient conditions to guarantee that an R-algebra is a Jacobson ring.

Proposition 2.4 ([3], Proposition 5)**.** Let R be a commutative, Jacobosn ring and A an R-algebra which satisfies the strong Nullstellensatz. Assume that for each maximal ideal $\mathfrak{m}$ of R, the algebra $(A/\mathfrak{m})[x]$ satisfies the Nullstellensatz over $k = R/\mathfrak{m}$. Then the Jacobson radical of A is nilpotent. In particular if A is Noetherian, then A is a Jacobson ring.

As we have seen, if one wants to verify the Nullstellensatz it is enough to check the condition of generic flatness. In the particular case of Ore extensions, we shall prove that these rings satisfy generic flatness. For that, let us recall its definition.

Definition 2.6 ([4]; [5], page 34)**.** Let A be a ring, σ a ring endomorphism of A, and δ a σ-derivation on A (δ is a homomorphism of abelian groups satisfying $\delta(a_1a_2) = \sigma(a_1)\delta(a_2) + \delta(a_1)a_2$, for every $a_1, a_2 \in A$). We shall write $B = A[x; \sigma, \delta]$ provided

(a) B is a ring, containing A as a subring;

(b) x is an element of B;

(c) B is a free left A-module with basis $\{1, x, x^2, \ldots\}$;

(d) $xa = \sigma(a)x + \delta(a)$, for all $a \in A$.

Such a ring B is called an *Ore extension* over A. If σ is injective, B is called an *Ore extension of injective type.*

When we have σ an automorphism, any element of B can be written either as a polynomial in x with all its coefficients on the left or as a polynomial with all its coefficients on the right. If A is an algebra over a commutative ring R, we want B to be an R-algebra as well. This will be the case if σ is an R-automorphism and δ vanishes on R.

We want to extend from the coefficient ring the property of generic flatness to the Ore extension. We will do it by guaranteeing that the extension is an algebra.

Proposition 2.5 ([3], Theorem 1)**.** Let R be a commutative domain, and let A be a right Noetherian R-algebra which satisfies right generic flatness. Let σ be an R-automorphism of A and δ a σ-derivation such that $\delta(R) = 0$. Then the Ore extension $B = A[x; \sigma, \delta]$ satisfies right generic flatness.

We can note that a Noetherian Ore extension with derivation zero in every element of a commutative domain satisfies generic flatness. This result is easy to extend in the following sense: Let R be a commutative Noetherian domain and A an R-algebra obtained from R by a finite succession of Ore extensions, each of which preserves the R-algebra structure. Then A satisfies generic flatness ([3], page 270). The following proposition is one of the most important results established in [3] about the Nullstellensatz and Jacobson's property. We include its proof with the aim of showing the importance of the arguments involved there.

Proposition 2.6 ([3], Theorem 2)**.** Let A be a finitely iterated Ore extension of the commutative Noetherian Jacobson ring R (field $\Bbbk$, respectively), which preserves the algebra structure. Then A satisfies the strong Nullstellensatz (the Nullstellensatz, respectively), and A is a Jacobson ring.

Proof. We follow the proof presented in [3]. If A is an R-algebra, the argument above on finite succession of Ore extensions guarantees that the assumptions of Proposition 2.3 hold, so A satisfies the strong Nullstellensatz.

Now, if A is an algebra over the field $\Bbbk$, the ring $A[x]$ can be viewed as an iterated Ore extension of $\Bbbk[x]$. Again, the argument on finite succession of Ore extensions implies that the hypotheses of Proposition 2.3 and Corollary 2.1 hold, so that A satisfies the Nullstellensatz.

In either case, we consider Proposition 2.4 since $A[x]$ is an iterated Ore extension of R or $\Bbbk$, whence $(A/M)[x]$ is an iterated Ore extension of $R/\mathfrak{m}$ and satisfies the Nullstellensatz. □

Since generic flatness and the strong Nullstellensatz are satisfied by remarkable families of finitely-generated Noetherian algebras, one might think that all finitely-generated Noetherian algebras satisfy these properties. However, this is not the case, as we can appreciate in the following example.

Example 2.1 ([3], Section IV)**.** Consider the ring $R = \mathbb{Z}[\frac{1}{2}, y, y^{-1}]$. Let T be the multiplicatively closed subset generated by the set of elements $\{y + 2n \mid n \in \mathbb{Z}\}$. Let $S = R_T$ be the corresponding localization. It is easy to see that S is a commutative, Noetherian, Jacobson ring, with an automorphism σ defined by $\sigma(y) = y + 2$. Consider the associated twisted group ring $A = S[x, x^{-1}; \sigma]$ which consists of polynomials in x and x^{-1} satisfying the relation $x^{-1}yx = y + 2$, and, in general, $x^{-n}yx^n = y + 2n$ and $x^{-n}y^{-1}x^n = (y + 2n)^{-1}$. Then A is a finitely-generated, Noetherian, Jacobson $\mathbb{Z}$-algebra which is a primitive ring, and hence A does not satisfy the strong Nullstellensatz or generic flatness ([3], Theorem 3).

A second approach to the Hilbert's Nullstellensatz in the non-commutative case was given by McConnell and Robson in [11]. This approach is an extended version of generic flatness to see how the theorem is satisfied in certain non-commutative affine algebras over a field. More precisely, in [11], p. 227, we can note that for certain non-commutative affine algebras R over a field $\Bbbk$, if the following properties hold:

(i) (Endomorphism property) For each simple right R-module M, $\text{End}(M)$ is algebraic over $\Bbbk$, and,

(ii) (Radical property) the Jacobson radical of each factor ring of R is nilpotent,

then, it is said that R *satisfies the Nullstellensatz over* $\Bbbk$.

Next, we recall the definition of generic flatness given by McConnell and Robson [11] over a commutative integral domain D (compare with Definition 2.5).

Definition 2.7 ([11], Definition 1)**.** We say that a D-algebra R is *generically flat* over D, if for each finitely generated right R-module M_R, there exists $0 \neq d \in D$ such that M_d is free over D_d.

There is a well-known connection between Definition 2.7 and the Nullstellensatz as we can see in [10], or Proposition 2.3 and Corollary 2.1.

About the notions of generically flatness, the endomorphism property and the Nullstellensatz, in [11, Lemmas 2 and 3], we can find some interesting relations: (i) Let R be a $\Bbbk$-algebra and x be a central indeterminate. If $R[x]$ is generically flat over $\Bbbk[x]$ then R has the endomorphism property. (ii) If R is a $\Bbbk$-algebra, x is a central indeterminate and $R[x]$ has the endomorphism property then R satisfies the Nullstellensatz. (iii) If R is a $\Bbbk$-algebra such that $R[x, y]$ is generically flat over $\Bbbk[y]$ then R satisfies the Nullstellensatz.

The following notion extends Definition 2.7.

Definition 2.8 ([11], Definition 5)**.** A D-algebra R is $(\mathbb{N}, \mathbb{N})-$*generically flat* over D, if $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is generically flat over $D[y_1, \ldots, y_m]$ for all $n, m \in \mathbb{N}$.

Lemma 2.1 ([11], Lemma 6)**.** Let $R \subseteq S$ be D-algebras and let R be $(\mathbb{N}, \mathbb{N})$-generically flat over D. Suppose that either one of the following conditions holds:

(i) S is a finite extension of R (i.e. S is finitely generated as a right R-module);

(ii) S is generated over R by an element z such that $zR = Rz$.

Then S is $(\mathbb{N}, \mathbb{N})$-generically flat over D.

Proof. Let us see the two cases.

(i) We follow the proof presented in [11]. If we have that the algebraic structure $S[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a finitely generated $R[x_1, \ldots, x_n, y_1, \ldots, y_m]$-module, then it is enough to prove that S is generically flat over D. Neverthless, any finitely generated right S-module is also finitely generated as a right R-module, so we are done.

(ii) It is enough to show that S is generically flat over D, so by [6], page 349, we consider a cyclic S-module M, say $M \cong S/I$, with I a right ideal of S. If one defines, for every n

$$I_n = \{r \in R \mid rz^n \in z^{n-1}R + \cdots + zR + R + I\},$$

then one obtains a chain of $R-$modules

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n \subseteq \cdots \subseteq M = \bigcup M_n,$$

where $M_n = \left(I + \sum_{i=0}^{n-1} Rz^i\right)/I$ and $M_{n+1}/M_n \cong R/I_n$ as a D-module. Now, let $N = R[x]/\sum x^n I_n$ where x is a central indeterminate. N is a cyclic $R[x]-$module and so, by hypothesis, N_d is free over D_d for some $0 \neq d \in D$, here the condition $(\mathbb{N}, \mathbb{N})-$generic flatness is needed). Thus each D_d-direct summand $(R/I_n)_d$ of N_d is projective and hence M_d splits, that is, $M_d \cong \oplus(R/I_n)_d \cong N_d$. Therefor M_d is free.

□

The following result establishes that for a filtered ring, the $(\mathbb{N}, \mathbb{N})$-generically flat property inherits from its associated graded ring (see [6] for more details about filtered and graded rings).

Lemma 2.2 ([11], Lemma 7)**.** Let S be a filtered D-algebra with $D \subseteq S_0$ and suppose that the associated graded ring $\mathrm{Gr}(S)$ is $(\mathbb{N}, \mathbb{N})$-generically flat over D. Then so too is S.

Proof. It is sufficient to show that S is generically flat. If M is a finitely generated right S-module, it can be filtered so that $\mathrm{Gr}(M)$ is finitely generated over $\mathrm{Gr}(S)$. Therefore $(\mathrm{Gr}(M))_d$ is free over D_d for some d, and so, arguing as in Lemma 2.1 (ii), $M_d \cong (\mathrm{Gr}(M))_d$ and thus it is free. □

In [11] it was shown that a kind of non-commutative rings are $(\mathbb{N}, \mathbb{N})$-generically flat and so they satisfy the *Nullstellensatz*.

Definition 2.9 ([11], Definition 8)**.** Let R, S be k-algebras (k a commutative ring) with $R \subset S$. S is an *almost extension* (also *almost commutative algebra*) of R, if the following conditions hold:

(i) S is generated over R by a finite set of elements $\{x_1, \ldots, x_n\}$;

(ii) $x_i R + R = R x_i + R$;

(iii) $x_i x_j - x_j x_i \in \sum_{k=1}^{n} R x_k + R$.

Lemma 2.3 ([11], Lemma 9). If R is $(\mathbb{N}, \mathbb{N})$-generically flat over D and S is an almost normalizing extension of R then S is $(\mathbb{N}, \mathbb{N})$-generically flat over D.

In [3] and [11] the definition of generically flat and generically free are used as the same. However, the two notions are not equivalent (recall that every free module is flat but the converse is not true).

A third approach to the subject was presented by Zhang et al. in [12]. In that paper, an *R-module M is generically flat over a domain R if there is a simple localization R_s such that $M_s = M \otimes_R R_s$ is flat over R_s.* Generically projective and generically free modules are defined similarly.

A technique to verify that the modules over a fixed ring are generically free is to check that all the associated graded modules are generically projective as the following proposition establishes.

Proposition 2.7 ([12], Proposition 3.8). Let R be a commutative domain and let $A = \bigcup F_n$ be an $\mathbb{N}$-filtered R-algebra. If every finite graded right $\mathrm{Gr}(A)$-module is generically projective over R, then every finite right A-module is generically free over R.

We recall the results of the previous subsections in the following lemma, in which, part (i) was established above and the part (ii) is a special case of Proposition 2.4.

Lemma 2.4 ([12], Lemma 3.9). Let A be a right Noetherian algebra over a field $\Bbbk$.

(i) If every simple right $A[x]$-module is generically free over $\Bbbk[x]$, then A satisfies the Nullstellensatz.

(ii) If $A[x]$ satisfies the Nullstellensatz, then A is a Jacobson algebra.

With these new tools, we can give another approach of Nullstellensatz over the Ore extensions.

Proposition 2.8 ([12], Theorem 3.10)**.** Let R be a commutative domain. Let A be a right Noetherian R-algebra such that every finite right A-module is generically free over R. Let $A[x; \sigma, \delta]$ be an Ore extension for an R-linear automorphism σ and a R-linear σ-derivation δ. Then every finite right $A[x; \sigma, \delta]$-module is generically free over R.

In order to state a result that helps us to describe the Hilbert's Nullstellensatz, let us remind that an $\mathbb{N}$-graded R-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ is called *locally finite*, if each homogeneous component A_i is a finite R-module for every i.

The following result is one of the most important theorems established in [12]. This will be very important in the next section.

Proposition 2.9 ([12], Theorem 0.4)**.** Let A be an $\mathbb{N}$-filtered algebra over a field $\Bbbk$ whose associated graded ring is locally finite and right Noetherian. Then A is a Jacobson algebra which satisfies the Nullstellensatz.

Remark 2.1 ([13], Proposition 2.10)**.** Let A be a $\Bbbk$-algebra. A is finitely graded if and only if there exists a graded isomorphism of $\Bbbk$-algebras $A \cong \Bbbk\{x_1, \dots, x_n\}/I$, where $\Bbbk\{x_1, \dots, x_n\}$ denotes the free algebra over $\Bbbk$ and I is a proper homogeneous two-sided ideal of $\Bbbk\{x_1, \dots, x_n\}$. In such case, A is locally finite, i.e., for every $n \in \mathbb{N}$, $\dim_{\Bbbk} A_n < \infty$.

3 Examples

In this section, we illustrate with several examples the results established in the previous section. With this aim we will consider the family of non-commutative rings known as skew PBW extensiones. The reason for this choice is the attention that these objects have had in recent years (c.f. [14],[15],[16],[17],[18],[19],[20],[21] and [22]).

3.1 Skew Poincaré-Birkhoff-Witt extensions

Skew PBW (PBW denotes Poincaré-Birkhoff-Witt) extensions were defined in [23] as a generalization of PBW extensions introduced in [24]. During the last years several ring and homological properties of these extensions have been studied (e.g., [25],[26],[27],[28],[29] and [30]). For details about relations between these extensions and other non-commutative rings of polynomial type, see [31] or [32]. In particular, skew PBW extensions extend Ore extensions of injective type (as a matter of fact, in [33], Example 1, there are examples of skew PBW extensions which cannot be expressed as Ore extensions).

Definition 3.1 ([23], Definition 1)**.** We say that A is a *skew PBW extension* of R (also called *σ-PBW extension of* R), which is denoted by $A := \sigma(R)\langle x_1, \dots, x_n\rangle$, if the following conditions hold:

(i) R is a subring of A sharing the same multiplicative identity element.

(ii) There exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements

$$\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}(x^0 := 1);$$

(iii) For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$;

(iv) For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Proposition 3.1 ([23], Proposition 3)**.** Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that $x_i r = \sigma_i(r)x_i + \delta_i(r)$, for each $r \in R$.

Proof. For every $1 \leq i \leq n$ and each $r \in R$ we have elements $c_{i,r}, r_i \in R$ such that $x_i r = c_{i,r} x_i + r_i$; since $\mathrm{Mon}(A)$ is a R-basis of A $c_{i,r}$ and r_i are unique for r, so we define $\sigma_i, \delta_i : R \to R$ by $\sigma_i(r) = c_{i,r}$, $\delta_i(r) = r_i$. We can check that σ_i is a ring endomorphism and δ_i is a σ_i-derivation of R. For $r, s \in R$ we have that

$$\begin{aligned} x_i(r+s) &= \sigma_i(r+s)x_i + \delta_i(r+s) \\ x_i r + x_i s &= \sigma_i(r)x_i + \delta_i(r) + \sigma_i(s)x_i + \delta(s) \\ &= (\sigma_i(r) + \sigma_i(s))x_i + \delta_i(r) + \delta_i(r), \end{aligned}$$

so we have that $\sigma_i(r+s) = \sigma_i(r) + \sigma_i(s)$ and $\delta(r+s) = \delta_i(r) + \delta_i(r)$, and

$$\begin{aligned} x_i(rs) &= \sigma_i(rs)x_i + \delta_i(rs) \\ (x_i r)s &= (\sigma_i(r)x_i + \delta_i(r))s \\ &= \sigma_i(r)x_i s + \delta_i(r)s \\ &= \sigma_i(r)(\sigma_i(s)x_i + \delta_i(s)) + \delta_i(r)s \\ &= \sigma_i(r)\sigma_i(s)x_i + \sigma_i(r)\delta_i(s) + \delta_i(r)s, \end{aligned}$$

we have $\sigma_i(rs) = \sigma_i(r)\sigma_i(s)$ and $\delta_i(rs) = \sigma_i(r)\delta_i(s) + \delta_i(r)s$ (this is de condition of σ_i-derivation), we can note also that $x_i = x_i 1 = \sigma_i(1)x_i + \delta_i(1)$, so $\sigma_i(1) = 1$ and $\delta_i(1) = 0$. Moreover, by the Definition 3.1 (iii), $c_{i,r} \neq 0$ for $r \neq 0$. This means that σ_i is injective. □

Definition 3.2 ([23], Definition 4)**.** Let A be a skew PBW extension.

(a) A is quasi-commutative, if the conditions (iii) and (iv) in the Definition 3.1 are replaced by the following:

- (iii') For each $1 \leq i \leq n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$.
- (iv') For any elements x_i, x_j with $1 \leq i, j \leq n$, there exists an element $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$.

(b) A is bijective, if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i, j \leq n$.

Skew PBW extensions are filtered rings, as the following proposition shows.

Proposition 3.2 ([31], Theorem 2.2)**.** Let A be an arbitrary skew PBW extension of R. Then, A is a filtered ring with filtration given by

$$F_m := \begin{cases} R, & \text{if } m = 0 \\ \{f \in A \mid \deg(f) \leq m\}, & \text{if } m \geq 1. \end{cases}$$

and the corresponding graded ring $\mathrm{Gr}(A)$ is a quasi-commutative skew PBW extension of R. Moreover, if A is bijective, then $\mathrm{Gr}(A)$ is a quasi-commutative bijective skew PBW extension of R.

Proposition 3.3 ([31], Theorem 2.3)**.** Let A be a quasi-commutative skew PBW extension of a ring R. (i) A is isomorphic to an iterated skew polynomial ring of endomorphism type. (ii) If A is bijective, then each endomorphism is bijective.

Corollary 3.1 ([31], Corollary 2.4. Hilbert Basis Theorem)**.** Let A be a bijective skew PBW extension of R. If R is a left Noetherian ring, then A is also a left Noetherian ring.

The following are examples of skew PBW extensions (see [3] and [11] for a detailed definition of every algebraic structure): classical polynomial rings, skew polynomial rings of derivation type, the Weyl algebra, the universal enveloping algebra of a finite dimensional Lie algebra, the Woronowicz algebra, the q-Heisenberg algebra, the additive analogue of the Weyl algebra, the multiplicative analogue of the Weyl algebra. For all these examples we know that the Hilbert's Nullstellensatz holds (see [31]).

Recall from Definition 2.4 that a $\Bbbk$-algebra A satisfies the Nullstellensatz, if for any simple right A-module M, the division ring $\mathrm{End}_A(M)$ is algebraic over $\Bbbk$. This will be our standard version of Hilbert's Nullstellensatz that the following result extends for skew PBW extensions.

Proposition 3.4. Let R be a commutative domain. Let B be a right Noetherian R-algebra such that every finite right B-module is generically free over R. Let A be a skew PBW extension of B for an R-linear automorphism σ_i and a R-linear σ_i-derivation δ_i, for $1 \leq i \leq n$ Then every finite right A-module is generically free over R.

Proof. Let A be a skew PBW extension of B, with B a right Noetherian R-algebra such that every finite right B-module is generically free over R. By Proposition 3.2 we know that $\mathrm{Gr}(A)$ is isomorphic to $B[z_1; \theta_1] \cdots [z_n; \theta_n]$, and we have that θ_i is a R-lineal automorphism, by hypothesis every finite right $B-$module is generically free over R, whence Proposition 2.8 guarantees that every finite right $B[z_1; \theta_1]-$module is generically free over R, so we can conclude that every finite right $B[z_1; \theta_1] \cdots [z_1, ; \theta_n]$-module is generically free and by Proposition 2.7 we have that every finite right A-module is generically free over R. □

Following Proposition 2.9 we can give conditions that are sufficient for a skew PBW extension to satisfy Hilbert's Nullstellensatz.

Theorem 3.1. Let A be a bijective skew PBW extension of a Noetherian $\Bbbk$-algebra R such that A is also a $\Bbbk$-algebra and $\mathrm{Gr}(A)$ is locally finite. Then A is a Jacobson algebra which satisfies the Nullstellensatz.

Proof. By Proposition 3.1, since A is a bijective skew PBW extension of R Noetherian, A is left Noetherian. According to Proposition 3.2, $\mathrm{Gr}(A)$ is a quasi-commutative bijective skew PBW extension isomorphic to an iterated skew polynomial ring $R[z_1;\theta_1][z_2;\theta_2]\cdots[z_n;\theta_n]$ such that each θ_i is bijective, $1 \leq i \leq n$ and by the Proposition 3.3 $\mathrm{Gr}(A)$ is left Noetherian, and by the hypothesis $\mathrm{Gr}(A)$ is locally finite. So we have A $\mathbb{N}$-filtered algebra over $\Bbbk$ (Proposition 3.2) whose associated graded ring is locally finite and left Noetherian. Then, by Proposition 2.9, A is a Jacobson algebra which satisfies the Nullstellensatz. □

Corollary 3.2. Every bijective skew PBW extension which preserves the $\Bbbk$-algebra structure whose associated graded ring is finitely graded is a Jacobson algebra which satisfies the Nullstellensatz

Proof. If we have A skew PBW extension over $\Bbbk$ whose associated graded ring $\mathrm{Gr}(A)$ is finitely graded, by Remark 2.1. is locally finite and by Proposition 3.1 is left Noetherian. Then from the Theorem 3.1 A is a Jacobson algebra which satisfies the Nullstellensatz. □

In the next section we will illustrate the Nullstellensatz with some particular examples of skew PBW extensions. As a matter of fact, ring and homological properties of these examples have been studied by several authors (c.f. [34],[35],[32],[33], [21],[36] and [13]).

3.1.1 Classical PBW extensions

Example 3.1 (Classical polynomial rings)**.** Let $\Bbbk[x_1,\dots,x_n]$ be the polynomial ring with $\Bbbk$ a field. The polynomial ring is an Ore extension with $\sigma_i = i_{\Bbbk[x_1,\dots,x_n]}$ and $\delta_i = 0$, for $1 \leq i \leq n$, Therefore, we have an extension over a Noetherian ring $\Bbbk$ which preserves the algebra structure.

Then, polynomial ring satisfies the hypothesis of Proposition 2.6, whence the Nullstellensatz holds and it is a Jacobson algebra.

We can note also that $\Bbbk[x_1, \dots, x_n]$ is an $\mathbb{N}$-filtered A algebra over $\Bbbk$ and its associated graded is $\Bbbk[x_1, \dots, x_n]$. We know that $\Bbbk[x_1, \dots, x_n]$ is Noetherian and locally finite. Then, for the Proposition 2.9 $\Bbbk[x_1, \dots, x_n]$ is a Jacobson algebra that satisfies the Nullstellensatz.

Finally, it is clear that the polynomial ring is a skew PBW extension. Since $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = 0$, for any $r \in \Bbbk$ and $1 \leq i, j \leq n$. The $\Bbbk$-free basis is $\mathrm{Mon}(\Bbbk[x_1, \dots, x_n])$. Every skew PBW extension is a filtered ring and its associated graded is locally finite. By Theorem 3.1, $\Bbbk[x_1, \dots, x_n]$ is a Jacobson algebra that satisfies the Nullstellensatz.

Example 3.2 (Universal enveloping algebra of a Lie algebra)**.** Let $\Bbbk$ be a field and $\mathcal{G}$ a finite dimensional Lie algebra over $\Bbbk$ with basis $\{x_1 \dots, x_n\}$. The **universal enveloping algebra of** $\mathcal{G}$, denoted by $\mathcal{U}(\mathcal{G})$, is not necessarily an Ore extension, and so we cannot conclude the Nustellensatz holds using Proposition 2.6. Now, in [37], page 75, it was shown that $\mathcal{U}(\mathcal{G})$ is an $\mathbb{N}$-filtered algebra. In [38], page 30, we can see that its associated graded is isomorphic to the classical polynomial ring. Therefore, this algebra is Noetherian and locally finite. Due to Proposition 2.9, $\mathcal{U}(\mathcal{G})$ is a Jacobson algebra that satisfies the Nullstellensatz.

The universal enveloping algebra of $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$, can be seen as a skew PBW extension. In [31], page 1211, the authors shown that there exists a skew PBW extension $A = \sigma(\Bbbk)\langle x_1, \dots, x_n\rangle$ such that $\mathcal{U}(\mathcal{G}) \cong A$. Since $x_ir - rx_i = 0$ and $x_ix_j - x_jx_i = [x_i, x_j] = \Bbbk + \Bbbk x_1 + \cdots + \Bbbk x_n$. In this case, A is an $\mathbb{N}$-filtered algebra and its associated graded is isomorphic to the classical polynomial ring and is locally finite. By Theorem, 3.1 A is a Jacobson algebra that satisfies the Nullstellensatz.

3.1.2 3-dimensional skew polynomial algebras Following [39], Definition C4.3, 3-dimensional skew polynomial algebras $\mathcal{A}$ are $\Bbbk$-algebras generated by the indeterminates x, y, z restricted to relations $yz - \alpha zy = \lambda$, $zx - \beta xz = \mu$, and $xy - \gamma yx = \nu$, such that the following conditions hold: (i) $\lambda, \mu, \nu \in \Bbbk + \Bbbk x + \Bbbk y + \Bbbk z$, and $\alpha, \beta, \gamma \in \Bbbk \setminus \{0\}$; (ii) standard monomials, $\{x^iy^jz^k \mid i, j, k \geq 0\}$, are a $\Bbbk$-basis of the algebra.

From the definition, we can see that these algebras are skew PBW extensions; in fact, in [34] the authors shown that 3-dimensional skew polynomial algebras have a PBW basis and with the relations defined, we can note that these algebras satisfy the condition (iii) and (iv) of Definition 3.1. Note that 3 dimensional skew polynomials algebras are not necessarily PBW extensions in the sense of [24]. There exists a classification of 3-dimensional skew polynomial algebras, provided by [39], Theorem C.4.3.1. More precisely, up isomorphism, $\mathcal{A}$ is one of the following algebras:

(a) if $|\{\alpha, \beta, \gamma\}| = 3$, then $\mathcal{A}$ is defined by the relations $yz - \alpha zy = 0$, $zx - \beta xz = 0$, and $xy - \gamma yx = 0$.

(b) if $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma = 1$, then $\mathcal{A}$ is one of the following algebra:

   (i) $yz - zy = z$, $zx - \beta xz = y$, $xy - yx = x$;
   (ii) $yz - zy = z$, $zx - \beta xz = b$, $xy - yx = x$;
   (iii) $yz - zy = 0$, $zx - \beta xz = y$, $xy - yx = 0$;
   (iv) $yz - zy = 0$, $zx - \beta xz = B$, $xy - yx = 0$;
   (v) $yz - zy = az$, $zx - \beta xz = 0$, $xy - yx = x$;
   (vi) $yz - zy = z$, $zx - \beta xz = 0$, $xy - yx = 0$,

   where a, b are any elements of k. All nonzero values of b yield isomorphic algebras.

(c) If $|\{\alpha, \beta, \gamma\}| = 2$ and $\beta \neq \alpha = \gamma \neq 1$, then $\mathcal{A}$ is one of the following algebra:

   (i) $yz - \alpha zy = 0$, $zx - \beta xz = y + b$, and $xy - \gamma yx = 0$;
   (ii) $yz - \alpha zy = 0$, $zx - \beta xz = b$, and $xy - \gamma yx = 0$.

   In this case, $b \in k$ is an arbitrary element and, like before, any nonzero values of b give isomorphic algebras.

(d) If $\alpha = \beta = \gamma \neq 1$, then $\mathcal{A}$ is the algebra defined by the relations $yz - \alpha zy = a_1 x + b_1$, $zx - \beta xz = a_2 y + b_2$, and $xy - \gamma yx = a_3 z + b_3$. If $a_i = 0$ for $i = 1, 2, 3$, then all nonzero values of b_i give isomorphic algebras.

(e) If $\alpha = \beta = \gamma = 1$, then $\mathcal{A}$ is isomorphic to one of the following algebras:

(i) $yz - zy = x$, $zx - xz = y$, $xy - yx = z$;

(ii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = z$;

(iii) $yz - zy = 0$, $zx - xz = 0$, $xy - yx = b$;

(iv) $yz - zy = -y$, $zx - xz = x + y$, $xy - yx = 0$;

(v) $yz - zy = az$, $zx - xz = z$, $xy - yx = 0$;

With $a, b \in k$ arbitrary, and all nonzero values of b generate isomorphic algebras.

Some of these algebras are not Ore extensions. For this reason we cannot use Proposition 2.6.

3-dimensional skew polynomial algebras $\mathcal{A}$ are skew PBW extension as we remarked. From Proposition 3.2 we have that $\mathcal{A}$ is $\mathbb{N}$-filtered. The associated graded of this algebra is $\Bbbk_q[x, y, z]$ with q defined by an automorphism and is Noetherian and locally finite. By Proposition 2.9 or Theorem 3.1, the 3-dimensional skew polynomial algebras satisfy the Nullstellensatz.

Example 3.3 (Dispin algebra $\mathcal{U}(osp(1, 2))$)**.** Dispin algebra $\mathcal{U}(osp(1, 2))$, defined in [39], Definition C4.1, is the enveloping algebra of the Lie superalgebra $osp(1, 2)$. It is generated by the indeterminates x, y, z over a field $\Bbbk$ satisfying the relations $yz - zy = z, zx + xz = y, xy - yx = x$. This algebra is not an Ore extensions. Hence, we can not use Proposition 2.6 to conclude the Nullstellensatz.

We can note that $\mathcal{U}(osp(1, 2))$ is a skew PBW extension over $\Bbbk$; in [31], page 1215, we note that $\mathcal{U}(osp(1, 2)) \cong \sigma(\Bbbk)\langle x, y, z\rangle$, its associated graded is $\Bbbk_q[x, y, z]$ which is Noetherian and locally finite (indeed is a $\Bbbk$-algebra finitely graded). Then, due to Proposition 2.9 or Theorem 3.1 $\mathcal{U}(osp(1, 2))$ is a Jacobson algebra in which the Nullstellensatz holds.

Example 3.4 (Woronowicz algebra $\mathcal{W}_{\mathcal{V}}(\mathfrak{sl}(2, k))$)**.** This algebra was introduced by Woronowicz in [40] and redefined in [39] over an arbitrary field $\Bbbk$. It is generated by the indeterminates x, y, z subject to the relations $xz - v^4zx = (1 + v^2)x$, $xy - v^2yx = vz$, $zy - v^4yz = (1 + v^2)y$, where

$v \in \mathbb{k} \setminus \{0\}$ is not a root of unity. Since this algebra is not an Ore extension, we can not use Proposition 2.6 to conclude the Nullstellensatz.

On the other hand, since $\mathcal{W}_{\mathcal{V}}(\mathfrak{sl}(2,k))$ is a skew PBW extension over $\mathbb{k}$, i.e., $\mathcal{W}_{\mathcal{V}}(\mathfrak{sl}(2,k)) \cong \sigma(\mathbb{k})\langle x,y,z\rangle$, its associated graded is $\mathbb{k}_q[x,y,z]$ which is Noetherian and locally finite (indeed is a $\mathbb{k}$-algebra finitely graded). Then, due to Proposition 2.9 or Theorem 3.1, $\mathcal{W}_{\mathcal{V}}(\mathfrak{sl}(2,k))$ is a Jacobson algebra in which the Nullstellensatz holds.

3.1.3 Other examples

Example 3.5 (Multiplicative analogue of the Weyl algebra)**.** The $\mathbb{k}$-algebra $\mathcal{O}_n(\lambda_{ji})$, defined in [41], is generated by the indeterminates $x_1,\dots,x_n$ subject to the relations: $x_jx_i = \lambda_{ji}x_ix_j$ with $1 \le i < j \le n$, and $\lambda_{ji} \in \mathbb{k} \setminus \{0\}$.

We can note that $\mathcal{O}_n(\lambda_{ji})$ is not an Ore extension over $\mathbb{k}$, but, $\mathcal{O}_n(\lambda_{ji})$ is an Ore extension over $\mathbb{k}[x_1]$ (see [38, page 29]). This is a finitely iterated Ore extension $(\mathbb{k}[x_1][x_2;\sigma_2]\cdots[x_n;\sigma_n]$ with $x_jx_i = \sigma_j(x_i)x_j = \lambda_{ji}x_ix_j)$ and we have that $\mathbb{k}[x_1]$ is a Jacobson ring, and preserves the algebra structure i.e. $\sigma_i(k) = k$, for all $k \in \mathbb{k}$ and we have that $\delta_i = 0$. Then, due to Proposition 2.6, $\mathcal{O}_n(\lambda_{ji})$ satisfies the Nullstellensatz .

We can note for the relations that $\mathcal{O}_n(\lambda_{ji})$ the condition (iii) of the Definition 3.1 and the condition (iv) ($\delta = 0$). In [31], we can note that $\mathcal{O}_n(\lambda_{ji}) \cong A = \sigma(\mathbb{k})\langle x_1,\dots,x_n\rangle$. A is $\mathbb{N}$-filtered and its associated graded is $\mathbb{k}_q[x_1,\dots,x_n]$, its associated graded is Noetherian and locally finite. Then, by Proposition 2.9 or Theorem 3.1 $\mathcal{O}_n(\lambda_{ji})$ satisfies Nullstellensatz.

Example 3.6 (Additive analogue of the Weyl algebra)**.** The $\mathbb{k}$-algebra $A_n(q_1,\dots,q_n)$ is the algebra generated by the indeterminates $x_1,\dots,x_n$, $y_1,\dots,y_n$ subject to the relations $x_jx_i = x_ix_j$ and $y_jy_i = y_iy_j$ for $1 \le i,j \le n$, $y_ix_j = x_jy_i$ for $i \ne j$ and $y_ix_i = q_ix_iy_i + 1$ for $1 \le i \le n$, where $q_i \in \mathbb{k} \setminus \{0\}$.

$A_n(q_1,\dots,q_n)$ is not an Ore extension over $\mathbb{k}$. In [38], the authors proved that this algebra is an Ore extension over $\mathbb{k}[x_1,\dots,x_n]$, i.e. $\mathbb{k}[x_1,\dots,x_n][y_1;\sigma_1,\delta_1]\cdots[y_n;\sigma_n,\delta_n]$ with $y_ix_i = \sigma_i(x_i)y_i + \delta_i(x_i) = q_ix_iy_i + 1$. Here, we have, for the Example 3.1 that $\mathbb{k}[x_1,\dots,x_n]$ is a Jacobson ring and is commutative Noetherian ring. Then, due to Proposition 2.6 $A_n(q_1,\dots,q_n)$ satisfies the Nullstellensatz.

From the relations that describe the algebra, in [31] the authors shown that $A_n(q_1, \dots, q_n) \cong \sigma(\Bbbk)\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$. Hence, the additive analogue of a Weyl algebra is a skew PBW extension. This algebra is $\mathbb{N}$-filtered and its associated graded is $\Bbbk_q[x_1, \dots, x_n, y_1, \dots, y_n]$ which is Noetherian and locally finite. Then, by Proposition 2.9 or Theorem 3.1, $A_n(q_1, \dots, q_n)$ satisfies Nullstellensatz.

Example 3.7 (Quantum algebra)**.** The $\Bbbk$-algebra $\mathcal{U}'(so(3, k))$, developed in [42] and [43] is generated by I_1, I_2, I_3 subject to relations $I_2I_1 - qI_1I_2 = -q^{1/2}I_3$, $I_3I_1 - q^{-1}I_1I_3 = q^{-1/2}I_2$, $I_3I_2 - qI_2I_3 = -q^{1/2}I_1$, where $q \in \Bbbk\setminus \{0\}$. This algebra is not a Ore extension. Then, we can not conclude Nullstellensatz with Proposition 2.6.

In [34] the authors proved that this algebra is a skew PBW extension $\mathcal{U}'(so(3, k)) \cong \sigma(\Bbbk)\langle I_1, I_2, I_3\rangle$. Then we have that the algebra is $\mathbb{N}$-filtered and its associated graded is $\Bbbk_q[I_1, I_2, I_3]$ which is Noetherian and locally finite. Then, due to Proposition 2.9 or Theorem 3.1, $\mathcal{U}'(so(3, k))$ satisfies Nullstellensatz.

Example 3.8 (q-Heisenberg algebra)**.** The $\Bbbk$-algebra $H_n(q)$ introduced in [44] is generated by the set of variables $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n$ subject to the relations:

$$\begin{aligned}
x_jx_i &= x_ix_j, & y_jy_i &= y_iy_j, & z_jz_i &= z_iz_j, & 1 \le i, j \le n,\\
z_jy_i &= y_iz_j, & z_jx_i &= x_iz_j, & y_jx_i &= x_iy_j & i \neq j,\\
z_iy_i &= qy_iz_i, & z_ix_i &= q^{-1}x_iz_i + y_i, & y_ix_i &= qx_iy_i, & q \in \Bbbk \setminus \{0\}, 1 \le i \le n,
\end{aligned}$$

In [38], the authors proved that this algebra is an Ore extension over the commutative ring $\Bbbk[x_1, \dots, x_n]$. Due to Proposition 2.6 we can conclude that $H_n(q)$ satisfies the Nullstellensatz.

On the other hand [31] we can see that $H_n(q)$ is a skew PBW extension isomorphic to $\sigma(\Bbbk)\langle x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n\rangle$. This algebra is $\mathbb{N}$-filtered and its associated graded is $\Bbbk_q[x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n]$ with q defined by the automorphisms σ_i and θ_i; its associated graded is Noetherian and locally finite. By Proposition 2.9 or Theorem 3.1, we conclude that $H_n(q)$ satisfies Nullstellensatz.

Example 3.9 (Jordan plane)**.** The $\Bbbk$-algebra $\mathcal{J}$ introduced in [45] is the free algebra defined by $\mathcal{J} := \Bbbk\{x, y\}/\langle yx - xy - y^2\rangle$. $\mathcal{J}$ is a skew PBW extension of $k[y]$ with the product given by $xy = yx - y^2$. This algebra is $\mathbb{N}$-filtered an its associated graded is $\Bbbk[x, y]$ that is Noetherian and locally finite. Proposition 2.9 or Theorem 3.1 imply that $H_n(q)$ satisfies Nullstellensatz.

Remark 3.1. In [13], Remark 2.11, we can find another examples of skew PBW extensions that are locally finite (c.f. [20],[35] and [30]).

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