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Lie Algebra Representation, Conservation Laws and Some Invariant Solutions for a Generalized Emden-Fowler Equation

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Abstract

All generators of the optimal algebra associated with a generalization of the Endem-Fowler equation are showed; some of them allow to give invariant solutions. Variational symmetries and the respective conservation laws are also showed. Finally, a representation of Lie symmetry algebra is showed by groups of matrices.

Keywords: Invariant solutions; Lie symmetry group; optimal system; Lie algebra classification, variational simmetries, conservation laws.

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Representación del álgebra de Lie, leyes de conservación y algunas soluciones invariantes para una ecuación de Emden-Fowler generalizada

Resumen
Se muestran todos los generadores del álgebra óptima asociados con una generalización de la ecuación de Endem-Fowler; algunos de ellos permiten dar soluciones invariantes. También se muestran las simetrías variacionales y las respectivas leyes de conservación. Finalmente, se muestra una representación del álgebra de simetría de Lie mediante grupos de matrices.

Palabras clave: Soluciones invariantes; grupo de simetrías de Lie; sistema óptimo; clasificación de álgebras de Lie; simetrías variacionales; Leyes de conservación.

1 Introduction

It is known that the class of Emden-Fowler equations $y_{xx} = Ax^n y^m$, such that A, m, n are real constants, have applications in physics, astronomy and chemistry [1],[2],[3],[4]. In [5] Polyanin and Zaitsev present a generalized Emden-Fowler equation $y_{xx} = Ax^n y^m (y_x)^l$ with A, m, n, l real constants. They proposed for this equation a big amount of solutions for multiple combinations of the parameters A, n, m, l. In the particular case in which $A = -2, n = 1, m = -2$ and $l = 3$, that is,

$$y_{xx} = -2xy_x^3 y^{-2}, \tag{1}$$

in [5], it is proposed an implicit parameterized solution for τ, as follows:

$$x = \tau \left(c_1\tau^{\nu} + c_2\tau^{-\nu} \right), \quad \text{with } y = \tau^2, \;\; \nu = 3, \tag{2}$$

where c_1, c_2 are arbitrary constants. The Lie symmetry group associated to this equation is presented by Arrigo in [6], however, the computations used to obtain this result are not given in detail (such Lie symmetry group is an 8-dimensional Lie group). In [6] also it was reduce (1) by means of the canonical variable transformation method with three of these symmetries, which are specifically:

$$-\Pi_2 = -\frac{x^2}{y^2}\frac{\partial}{\partial_x} + \frac{x}{y}\frac{\partial}{\partial_y}, \quad \Pi_5 = x\frac{\partial}{\partial_x} + 2y\frac{\partial}{\partial_y}, \quad \Pi_9 = y\frac{\partial}{\partial_y}.$$

These symmetries allow us to obtain the corresponding transformations of (1):

1. $rs''(r) + 2s'(r) = 0$ with $x = (\frac{r^2}{3s(r)})^{1/3}$ and $y = (3rs(r))^{\frac{1}{3}}$,
2. $r^2s''(r) + r^2(s'(r))^2 - 2 = 0$ with $x = e^s$ and $y = r$,
3. $s''(r) + 2r(s'(r))^3 + (s'(r))^2 = 0$ with $x = r$ and $y = e^s$.

Since the symmetry group of (1) is an 8-dimensional group and following the ideas of citas [7],[8],[9], we search for its algebraic characteristics and some invariant solutions of (1). In fact, the goal of this work is: *i*) to calculate the 8-dimensional Lie symmetry group in all detail, *ii*) to present the optimal algebra (optimal system) for (1), *iii*) to use some elements of the optimal algebra to propose invariant solutions for (1), *iv*) to construct the Lagrangian with which we could determine the variational symmetries and thus to present conservation laws associated, and finally *iv*) to classify the Lie algebra associated to (1) by groups of matrices.

2 Continuous group of Lie symmetries

The Lie symmetry group associated to (1) is an 8-dimensional Lie group presented by Arrigo in [6], however, the computations used to obtain this result are not given in detail. In this section we present the computational details of that result.

Proposition 2.1. The Lie symmetry group for the equation (1) is generated by the following vector fields:

$$\Pi_1 = 2yx^2\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}, \quad \Pi_2 = \frac{x^2}{y^2}\frac{\partial}{\partial x} - \frac{x}{y}\frac{\partial}{\partial y}, \quad \Pi_3 = 2xy^3\frac{\partial}{\partial x} + y^4\frac{\partial}{\partial y},$$

$$\Pi_4 = \frac{x}{y^3}\frac{\partial}{\partial x} - \frac{1}{y^2}\frac{\partial}{\partial y}, \qquad \Pi_5 = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \qquad \Pi_6 = y^2\frac{\partial}{\partial x},$$

$$\Pi_7 = \frac{1}{y}\frac{\partial}{\partial x}, \qquad \text{and} \qquad \Pi_9 = y\frac{\partial}{\partial y}. \tag{3}$$

Proof. The general form of a one-parameter Lie group which can be defined for (1), is given by $x \to x + \epsilon X(x,y) + ...$ *and* $y \to y + \epsilon Y(x,y) + ...$, where ϵ is the parameter of the group. The vector field associated to such group of transformations can be written as $\Gamma = X(x,y)\frac{\partial}{\partial x} + Y(x,y)\frac{\partial}{\partial y}$. Applying the second prolongation $\Gamma^{(2)} = \Gamma + Y_{[x]}\frac{\partial}{\partial y_x} + Y_{[xx]}\frac{\partial}{\partial y_{xx}}$ to (1), we can find the infinitesimal generators $X(x,y), Y(x,y)$ that satisfy the symmetry condition

$$y^2 Y_{[xx]} + 2yy_{xx}Y + 2y_x^3 X + 6xy_x^2 Y_{[x]} = 0, \tag{4}$$

where $Y_{[x]}, Y_{[xx]}$ are the coefficients in $\Gamma^{(2)}$ given by:

$$\begin{aligned} Y_{[x]} &= D_x[Y] - (D_x[X])y_x = Y_x + (Y_y - X_x)y_x - X_y y_x^2, \\ Y_{[xx]} &= D_x[Y_{[x]}] - (D_x[X])y_{xx} \\ &= Y_{xx} + (2Y_{xy} - X_{xx})y_x + (Y_{yy} - 2X_{xy})y_x^2 - X_{yy}y_x^3 \\ &+ (Y_y - 2X_x)y_{xx} - 3X_y y_x y_{xx}, \end{aligned} \tag{5}$$

being D_x the total derivative operator: $D_x = \partial_x + y_x\partial_y + y_{xx}\partial_{y_x} + \cdots$. After substituting (5) into (4) we get:

$$\begin{aligned} & y^2[Y_{xx} + (2Y_{xy} - X_{xx})y_x + (Y_{yy} - 2X_{xy})y_x^2 - X_{yy}y_x^3] \\ + \; & y^2[(Y_y - 2X_x)y_{xx} - 3X_y y_x y_{xx}] + 2yy_{xx}Y + 2y_x^3 X \\ + \; & 6xy_x^2[Y_x + (Y_y - X_x)y_x - X_y y_x^2] = 0. \end{aligned}$$

After substituting $y_{xx} = -2xy_x^3y^{-2}$ into the preceding expression and analysing the coefficients associated to y_x, we get the following determinant equations:

$$y^2 Y_{xx} = 0, \tag{6a}$$

$$2y^2 Y_{xy} - y^2 X_{xx} = 0, \tag{6b}$$

$$y^2 Y_{yy} - 2y^2 X_{xy} + 6xY_x = 0, \tag{6c}$$

$$2yX - y^3 X_{yy} - 2xyX_x + 4xyY_y - 4xY = 0. \tag{6d}$$

Solving (6a) we have:

$$Y = c_1(y)x + c_2(y), \tag{7}$$

where $c_1(y), c_2(y)$ are arbitrary functions. From (7), we get $Y_{xy} = c_1'(y)$, then writing this expression on (6b) we obtain that $X_{xx} = 2c_1'(y)$, therefore, integrating we have $X_x = 2c_1'(y)x + c_3(y)$, which implies $X = c_1'(y)x^2 + c_3(y)x + c_4(y)$, with $c_3(y), c_4(y)$ arbitrary functions. Substituting the previous expression and (7) into (6c), we obtain that $x(-3y^2c_1''(y) + 6c_1(y)) + y^2[c_2''(y)] - 2y^2[c_3'(y)] = 0$, then, taking the derivative with respect to x, we have $-3y^2c_1''(y) + 6c_1(y) = 0$, and solving this ODE for $c_1(y)$ we get $c_1(y) = k_1y^2 + \frac{k_2}{y}$, with k_1, k_2 arbitrary constants. Therefore,

$$Y = \left[k_1y^2 + \frac{k_2}{y}\right] x + c_2(y) \quad \text{and} \quad X = \left[2k_1y - \frac{k_2}{y^2}\right] x^2 + c_3(y)x + c_4(y).$$

Substituting the previous equation into (6d) we get $2yc_4(y) - y^3c_4''(y) + x\left(-y^3c_3''(y) + 4yc_2'(y) - 4c_2(y)\right) = 0$. Analyzing the coefficients with respect to x we have that $-y^3c_3''(y) + 4yc_2'(y) - 4c_2(y) = 0$, which in turn implies

$$c_3''(y) = \frac{4c_2'(y)}{y^2} - \frac{4c_2(y)}{y^3}, \tag{8}$$

and, consequently, $2yc_4(y) - y^3c_4''(y) = 0$. Thus, $c_4(y) = k_6y^2 + \frac{k_7}{y}$, with k_6, k_7 arbitrary constants. This allow us to conclude that X and Y are given by

$$Y = \left[k_1y^2 + \frac{k_2}{y}\right] x + c_2(y), \qquad X = \left[2k_1y - \frac{k_2}{y^2}\right] x^2 + c_3(y)x + k_6y^2 + \frac{k_7}{y}. \tag{9}$$

When we substitute the previous equations into (6c) we get:

$$2xk_1y^2 - 8y^2xk_1 + 6xk_1y^2 + \frac{2xk_2}{y} - \frac{8xk_2}{y} + \frac{6xk_2}{y} + c_2''(y)y^2 - 2y^2c_3'(y) = 0,$$

which is equivalent to $c_2''(y) - 2c_3'(y) = 0$. Then, integrating with respect to y we obtain

$$c_2'(y) - 2c_3(y) = k_9 \;\Rightarrow\; c_3(y) = \frac{c_2'(y) - k_9}{2} \;\Rightarrow\; \frac{c_2'''(y)}{2} = c_3''(y).$$

When substituting the last equation into (8), we get $-y^3\left(\frac{c_2'''(y)}{2}\right) + 4yc_2'(y) - 4c_2(y) = 0$ and solving for $c_2(y)$, it is obtained $c_2(y) = k_3y^4 + \frac{k_4}{y^2} + k_8y$, with k_3, k_4, k_8 arbitrary constants. Now, we know that $c_3(y) = \frac{c_2'(y)-k_9}{2}$, then $c_3(y) = 2k_3y^3 - \frac{k_4}{y^3} + \frac{k_8}{2} - \frac{k_9}{2}$ and, thus, X and Y from (9), have the following form:

$$X = \left(2k_1y + \frac{k_2}{y^2}\right)x^2 + \left(2k_3y^3 - \frac{k_4}{y^3} + \frac{k_8}{2} - \frac{k_9}{2}\right)x + k_6y^2 + \frac{k_7}{y},$$

$$Y = \left(k_1y^2 - \frac{k_2}{y}\right)x + k_3y^4 + k_8y + \frac{k_4}{y^2}.$$

Now, taking $k_5 = \frac{k_8}{2} - \frac{k_9}{2}$ and $K_4 = -k_4$ we get that the infinitesimal generators are:

$$\begin{aligned} X &= \left(2k_1y + \frac{k_2}{y^2}\right)x^2 + \left(2k_3y^3 + \frac{K_4}{y^3} + k_5\right)x + k_6y^2 + \frac{k_7}{y}, \\ Y &= \left(k_1y^2 - \frac{k_2}{y}\right)x + k_3y^4 + (2k_5 + k_9)y - \frac{K_4}{y^2}, \end{aligned} \tag{10}$$

with $k_1, \cdots K_4, \cdot, k_8$ arbitrary constants; this implies that the symmetry group is generated by the operators described in the statement of the proposition. □

3 Optimal algebra

Taking into account [10, 11, 12, 13], we present in this section the optimal algebra associated to the symmetry group of (1), that shows a systematic way to classify the invariant solutions.

To obtain the optimal algebra, we should first calculate the corresponding commutator table, which can be obtained from the operator

$$[\Pi_\alpha, \Pi_\beta] = \Pi_\alpha\Pi_\beta - \Pi_\beta\Pi_\alpha = \sum_{i=1}^{n} \left(\Pi_\alpha(\xi_\beta^i) - \Pi_\beta(\xi_\alpha^i)\right)\frac{\partial}{\partial x^i}, \tag{11}$$

where $i = 1, 2$, with $\alpha, \beta = 1, 2, 3, ..., 7, 9$ and $\xi_\alpha^i, \xi_\beta^i$ are the corresponding coefficients of the infinitesimal operators Π_α, Π_β. After applying the operator (11) to the symmetry group of (1), we obtain the operators that are shown in the following table

Table 1: Commutators table associated to the symmetry group of (1).

	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6	Π_7	Π_9
Π_1	0	0	0	$-3\Pi_2$	$-3\Pi_1$	$-\Pi_3$	$9\Pi_9 - 5\Pi_5$	$-\Pi_1$
Π_2	0	0	$-3\Pi_1$	0	$3\Pi_2$	$9\Pi_9 - 4\Pi_5$	$-\Pi_4$	$2\Pi_2$
Π_3	0	$3\Pi_1$	0	$3\Pi_5$	$-6\Pi_3$	0	$-3\Pi_6$	$-3\Pi_3$
Π_4	$3\Pi_2$	0	$-3\Pi_5$	0	$6\Pi_4$	$-3\Pi_7$	0	$3\Pi_4$
Π_5	$3\Pi_1$	$-3\Pi_2$	$6\Pi_3$	$-6\Pi_4$	0	$3\Pi_6$	$-3\Pi_7$	0
Π_6	Π_3	$-(9\Pi_9 - 4\Pi_5)$	0	$3\Pi_7$	$-3\Pi_6$	0	0	$-2\Pi_6$
Π_7	$-(9\Pi_9 - 5\Pi_5)$	Π_4	$3\Pi_6$	0	$3\Pi_7$	0	0	Π_7
Π_9	Π_1	$-2\Pi_2$	$3\Pi_3$	$-3\Pi_4$	0	$2\Pi_6$	$-\Pi_7$	0

Now, the next thing is to calculate the adjoint action representation of the symmetries of (1) and to do that, we use Table 1 and the operator

$$Ad(exp(\lambda\Pi))H = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}(ad(\Pi))^n G \text{ for the symmetries } \Pi \text{ and } G.$$

Making use of this operator, we can construct the Table 2, which shows the adjoint representation for each Π_i.

The following result shows what is the optimal algebra associated to the equation (1). The whole procedure will not be presented in this document due to its length.

Proposition 3.1. The optimal algebra associated to the equation (1) is given by the vector fields shown in the Table 3.

Table 2: Adjoint representation of the symmetry group of (1).

adj[]	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6	Π_7	Π_9
Π_1	Π_1	Π_2	Π_3	$\Pi_4 + 3\lambda\Pi_2$	$\Pi_5 + 3\lambda\Pi_1$	$\Pi_6 + \lambda\Pi_3$	$\Pi_7 - 9\lambda\Pi_9 + 5\lambda\Pi_5 + 3\lambda^2\Pi_1$	$\Pi_9 + \lambda\Pi_1$
Π_2	Π_1	Π_2	$\Pi_3 + 3\lambda\Pi_1$	Π_4	$\Pi_5 - 3\lambda\Pi_2$	$\Pi_6 - 9\lambda\Pi_9 + 4\lambda\Pi_5 + 3\lambda^2\Pi_2$	$\Pi_7 + \lambda\Pi_4$	$\Pi_9 - 2\lambda\Pi_2$
Π_3	Π_1	$\Pi_2 - 3\lambda\Pi_1$	Π_3	$\Pi_4 - 3\lambda\Pi_5 - 9\lambda^2\Pi_3$	$\Pi_5 + 6\lambda\Pi_3$	Π_6	$\Pi_7 + 3\lambda\Pi_6$	$\Pi_9 + 3\lambda\Pi_3$
Π_4	$\Pi_1 - 3\lambda\Pi_2$	Π_2	$\Pi_3 + 3\lambda\Pi_5 - 9\lambda^2\Pi_4$	Π_4	$\Pi_5 - 6\lambda\Pi_4$	$\Pi_6 + 3\lambda\Pi_7$	Π_7	$\Pi_9 - 3\lambda\Pi_4$
Π_5	$e^{-3\lambda}\Pi_1$	$e^{3\lambda}\Pi_2$	$e^{-6\lambda}\Pi_3$	$e^{6\lambda}\Pi_4$	Π_5	$e^{-3\lambda}\Pi_6$	$e^{3\lambda}\Pi_7$	Π_9
Π_6	$\Pi_1 - \lambda\Pi_3$	$\Pi_2 + 9\lambda\Pi_9 - 4\lambda\Pi_5 + 3\lambda^2\Pi_6$	Π_3	$\Pi_4 - 3\lambda\Pi_7$	$\Pi_5 + 3\lambda\Pi_6$	Π_6	Π_7	$\Pi_9 + 2\lambda\Pi_6$
Π_7	$\Pi_1 + 9\lambda\Pi_9 - 5\lambda\Pi_5 + 3\lambda^2\Pi_7$	$\Pi_2 - \lambda\Pi_4$	$\Pi_3 - 3\lambda\Pi_6$	Π_4	$\Pi_5 - 3\lambda\Pi_7$	Π_6	Π_7	$\Pi_9 - \lambda\Pi_7$
Π_9	$e^{-\lambda}\Pi_1$	$e^{2\lambda}\Pi_2$	$e^{-3\lambda}\Pi_3$	$e^{3\lambda}\Pi_4$	Π_5	$e^{-2\lambda}\Pi_6$	$e^{\lambda}\Pi_7$	Π_9

Table 3: Optimal algebra generators for (1).

<table>
<tr><th>Number of elements</th><th>Vector Fields</th><th>Number of elements</th><th>Vector Fields</th></tr>
<tr><td>1</td><td>Π_1</td><td rowspan="3">5</td><td rowspan="3">$\Pi_1+\Pi_2+\Pi_5+\Pi_7+\Pi_9$
$\Pi_2-2\Pi_4+\Pi_5+\Pi_6+\Pi_7$
$\Pi_1+\Pi_3+\Pi_4+\Pi_5+\Pi_9$
$\Pi_1+\Pi_2+\Pi_5+\Pi_6+\Pi_7$
$\Pi_1+\Pi_2+\Pi_3-\Pi_5+\Pi_9$
$\Pi_1+\Pi_2+\Pi_4+\Pi_5+\Pi_6$
$\Pi_2+\Pi_4-\Pi_6+\Pi_7+\Pi_9$
$\Pi_2+\Pi_4+\Pi_5+\Pi_6-\frac{1}{2}\Pi_7$
$\Pi_1+\Pi_2+\Pi_3+\Pi_5-\frac{1}{3}\Pi_9$
$\Pi_1+\Pi_4+\Pi_6+\Pi_7+\Pi_9$
$\Pi_2-3\Pi_4+\Pi_6+3\Pi_7+\Pi_9$
$\Pi_3-\frac{1}{3}\Pi_5+\Pi_6+\Pi_7+\Pi_9$
$\Pi_1-\frac{5}{3}\Pi_2+\Pi_3+\Pi_5+\Pi_9$
$\Pi_1+\Pi_3+\Pi_5+\Pi_6-\frac{3}{2}\Pi_9$
$\Pi_1+\Pi_3-\frac{1}{2}\Pi_5+\Pi_6+\Pi_9$
$\Pi_1+\Pi_4+\Pi_5+\Pi_6+\Pi_7$
$\Pi_2+2\Pi_4+\Pi_5+\Pi_6+\Pi_7$
$\Pi_1+\Pi_2+\Pi_6+\Pi_7+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_9$
$\Pi_1+\Pi_2+\Pi_4+\Pi_5-2\Pi_9$
$\Pi_1-\frac{5}{9}\Pi_2+\Pi_3+\Pi_4+\frac{5}{9}\Pi_5$
$\Pi_2+\Pi_3+\Pi_4+\Pi_7+\Pi_9$</td></tr>
<tr><td>2</td><td>$\Pi_1+\Pi_5$
$-\Pi_1+\Pi_3$
$\Pi_5+\Pi_7$
$\Pi_1+\Pi_3$
$\Pi_2+\Pi_4$</td></tr>
<tr><td>3</td><td>$\Pi_1+\Pi_2+\frac{5}{9}\Pi_5$
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$\Pi_1+\Pi_3+\Pi_6$
$\Pi_4+\Pi_6+\Pi_7$
$\Pi_1+\Pi_4+\Pi_7$
$\Pi_2+\Pi_4+\Pi_7$
$\Pi_3+\Pi_6+\Pi_7$
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$\Pi_1+\Pi_2+\Pi_3$
$\Pi_4+\Pi_5+\Pi_7$</td></tr>
<tr><td>4</td><td>$\Pi_1+\Pi_4+\Pi_7+\Pi_9$
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$\Pi_1+\Pi_2+\Pi_4+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3-\Pi_5$
$\Pi_1+\Pi_2+\Pi_3-\frac{3}{2}\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_6$
$\Pi_3+\Pi_6+\frac{2}{15}\Pi_7+\frac{9}{5}\Pi_9$
$\Pi_2+\Pi_3+\Pi_4+3\Pi_6$
$\Pi_1+\Pi_2+\Pi_5+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4$
$3\Pi_1-3\Pi_2+\Pi_3+\Pi_5$
$\Pi_1+\Pi_3+\Pi_5+\Pi_9$
$\Pi_3-\frac{2}{3}\Pi_5+\Pi_6+\Pi_9$
$\Pi_1+\Pi_3+\Pi_5-3\Pi_9$
$\Pi_5+\Pi_6+\Pi_7+\Pi_9$
$\Pi_2+\Pi_3+\Pi_4+\Pi_7$
$\Pi_1+\Pi_2+\Pi_5+\Pi_7$
$\Pi_1+\Pi_2+\Pi_4+\Pi_7$
$\Pi_1+\Pi_2+\Pi_3+\Pi_9$
$\Pi_1+\Pi_4+\Pi_6+\Pi_7$
$\Pi_2+\Pi_4+\Pi_6+\Pi_7$</td><td>6</td><td>$\Pi_1-2\Pi_2+\Pi_4+\Pi_7+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5-\frac{3}{2}\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5-3\Pi_9$
$\Pi_1+\Pi_2+\frac{50}{81}\Pi_5+\Pi_6+\Pi_7-\frac{5}{2}\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_7+\Pi_9$
$\Pi_1+\Pi_2+\Pi_4+\Pi_5+\Pi_7-2\Pi_9$
$\Pi_1-\frac{5}{3}\Pi_2+\Pi_3+\Pi_4+\Pi_5+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_6+\Pi_7$
$\Pi_1+\Pi_2+\Pi_3-\Pi_5+\Pi_7+\Pi_9$
$\Pi_2+\Pi_3-\Pi_5+\Pi_6+\Pi_7+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5+\Pi_9$
$\Pi_1+\Pi_4+\Pi_5+\Pi_6-\frac{3}{5}\Pi_7-\frac{9}{5}\Pi_9$
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<tr><td>7</td><td colspan="3">$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5+\Pi_6+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_5+\Pi_6+\Pi_7+\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\frac{1}{3}\Pi_4+\Pi_5+\Pi_7-\frac{3}{2}\Pi_9$
$\Pi_1-\Pi_3+\Pi_4+\Pi_5+\Pi_6+3\Pi_7-\frac{3}{2}\Pi_9$
$\Pi_1+\Pi_2+\Pi_4+8\Pi_5-\Pi_6+\Pi_7-15\Pi_9$
$\Pi_1+\Pi_2+\Pi_4-5\Pi_5-1\Pi_6+\Pi_7-9\Pi_9$
$\Pi_1+\Pi_3+\Pi_4+\Pi_5+\frac{3}{5}\Pi_6+\frac{3}{25}\Pi_7-3\Pi_9$
$\Pi_1+\Pi_2+\Pi_3+\Pi_4+\Pi_5+3\Pi_7-\frac{3}{2}\Pi_9$</td></tr>
</table>

4 Invariant solutions by some generators of the optimal algebra

In this section, we obtain invariant solutions taking into account some operators that generate the optimal algebra presented in Proposition 3.1. For this purpose, we use the method of invariant curve condition [11] (presented in section 4.3), which is given by the following equation

$$Q(x, y, y_x) = Y - y_x X = 0. \tag{12}$$

Using the element Π_1 from Proposition 3.1, under the condition (12), we obtain that $Q = Y_1 - y_x X_1 = 0$, which implies $(xy^2) - y_x(2yx^2) = 0$. After isolating y_x we get $y_x = \frac{y}{2x}$, then solving this ODE we have $|y(x)| = c\sqrt{|x|}$, where c is an arbitrary constant, which is an invariant solution for (1), using an analogous procedure with some of the elements of the optimal algebra (Table 3), we obtain both implicit and explicit invariant solutions that are shown in the Table 4, with c being an arbitrary constant.

Remark 4.1. The invariant solution of the numeral 4 from Table 4 is the same solution as the one presented by Arrigo in [6] (using the symmetry $-\Pi_2$,) if we use the method of invariant curve given in this document, nevertheless, the invariant solutions of numeral 1 and 2 shown in Table 4, are a new invariant solutions. The implicit solutions of the numerals $5, 6, 7, 8, 9$ and 10 from Table 4, are implicit solutions which are different from the implicit solutions (2) presented in [5].

Table 4: Solutions for (1) using invariant curve condition.

	Elements	$Q(x, y, y_x) = 0$	**Solutions**	**Type Solution**
1	Π_1	$xy^2 - y_x(2yx^2) = 0$	$\lvert y(x)\rvert = c\lvert x\rvert^{\frac{1}{2}}$	Explicit
2	$-\Pi_1 + \Pi_3$	$(-xy^2 + y^4) - y_x(-2yx^2 + 2xy^3) = 0$	$\lvert y(x)\rvert = c\lvert x\rvert^{-\frac{1}{2}}$	Explicit
3	$\Pi_5 + \Pi_7$	$2y - y_x(x + y^{-1}) = 0$	$y(x) = -\frac{3}{10x}$	Explicit
4	$\Pi_2 + \Pi_4$	$-(xy^{-1} + y^{-2}) - y_x(x^2y^{-2} + xy^{-3}) = 0$	$\lvert y(x)\rvert = c\lvert x\rvert^{-1}$	Explicit
5	$\Pi_1 + \Pi_5$	$(xy^2 + 2y) - y_x(2yx^2 + x) = 0$	$-3x^2y^2 + xy + 5 = 0$	Implicit
6	$\Pi_4 + \Pi_6 + \Pi_9$	$(-y^{-2} + y) - y_x(xy^{-3} + y^2) = 0$	$\frac{y^3-3}{y} = 2x.$	Implicit
7	$\Pi_5 + \Pi_6 + \Pi_7$	$(2y) - y_x(x + y^2 + y^{-1}) = 0$	$\frac{y^3-1}{y} = 3x.$	Implicit
8	$\Pi_3 + \Pi_6 + \Pi_7$	$(y^4) - y_x(2xy^3 + y^2 + y^{-1}) = 0$	$y^4 - y^2 = \frac{3}{2x}$	Implicit
9	$\Pi_3 + \Pi_6 + \frac{2}{15}\Pi_7 + \frac{9}{5}\Pi_9$	$(y^4 + 9y) - y_x\left(2xy^3 + y^2 + \frac{2y^{-1}}{15}\right) = 0$	$\frac{60y^4-43y^3+4}{(y+2y^7)} = 30x$	Implicit
10	$\Pi_5 + \Pi_6 + \Pi_7 + \Pi_9$	$(3y) - y_x(x + y^2 + y^{-1}) = 0$	$\frac{4y^3-5}{y} = 20x$	Implicit

5 Variational symmetries and conserved quantities

In this section, we present the variational symmetries of (1) and we are going to use them to define conservation laws via Noether's theorem [14]. First of all, we are going to determine the Lagrangian using the Jacobi Last Multiplier method, presented by Nucci in [15], and for this reason, we are urged to calculate the inverse of the determinant Δ,

$$\Delta = \begin{vmatrix} x & y_x & y_{xx} \\ \Pi_{5,x} & \Pi_{5,y} & \Pi_5^{(1)} \\ \Pi_{9,x} & \Pi_{9,y} & \Pi_9^{(1)} \end{vmatrix} = \begin{vmatrix} x & y_x & y_{xx} \\ x & 2y & y_x \\ 0 & y & y_x \end{vmatrix},$$

where $\Pi_{5,x}, \Pi_{5,y}, \Pi_{9,x}$, and $\Pi_{9,y}$ are the components of the symmetries Π_6, Π_9 shown in the Proposition 3 and $\Pi_5^{(1)}, \Pi_9^{(1)}$ as its first prolongations. Then we get $\Delta = \frac{2xy_x^3}{y}$ which implies that $M = \frac{1}{\Delta} = \frac{y}{2xy_x^3}$. Now, from [15], we know that M can also be written as $M = L_{y_x y_x}$ which means that $L_{y_x y_x} = \frac{y}{2xy_x^3}$, then integrating twice with respect to y_x we obtain the Lagrangian

$$L(x, y, y_x) = \frac{y}{4xy_x} + f_1(x, y)y_x + f_2(x, y), \tag{13}$$

where f_1, f_2 are arbitrary functions. From the preceding expression we can consider $f_1 = f_2 = 0$. It is possible to find more Lagrangians for (1) by considering other vector fields given in the Proposition 3. We then calculate

$$X(x, y)L_x + X_x(x, y)L + Y(x, y)L_y + Y_{[x]}(x, y)L_{y_x} = D_x[f(x, y)],$$

using (13) and (5). Thus we get

$$X\left(-\frac{y}{4x^2y_x}\right) + X_x\left(\frac{y}{4xy_x}\right) + Y\left(\frac{1}{4xy_x}\right) + \left(Y_x + (Y_y - X_x)y_x - X_y y_x^2\right)$$
$$\left(-\frac{y}{4xy_x^2}\right) - f_x - y_x f_y = 0.$$

From the preceding expression, rearranging and associating terms with respect to $1, y_x, y_x^2$ and y_x^3, we obtain the following determinant equations

$$Y_x = f_y = 0, \tag{14a}$$

$$yX_y - 4xf_x = 0, \tag{14b}$$
$$-yX + 2xyX_x + xY - xyY_y = 0. \tag{14c}$$

Solving the preceding system for X, Y and f we obtain the infinitesimal generators of Noether's symmetries

$$Y = a_1 y, \quad X = 4a_3 \ln(y) + a_2 x^{\frac{1}{2}}, \quad \text{and} \quad f(x) = a_3 \ln(x) + a_4. \tag{15}$$

with a_1, a_2, a_3 and a_4 arbitrary constants. Then, the Noether symmetry group or variational symmetries are

$$V_1 = y\frac{\partial}{\partial y}, \qquad V_2 = x^{\frac{1}{2}}\frac{\partial}{\partial x} \qquad \text{and} \qquad V_3 = \ln(y)\frac{\partial}{\partial x}. \tag{16}$$

According to [16], in order to obtain the conserved quantities or conservation laws, we should solve

$$I = (Xy_x - Y)L_{y_x} - XL + f,$$

using (13), (15) and (16). Therefore, the conserved quantities are given by

$$I_1 = \frac{y^2}{4xy_x^2} + a_3 \ln(x) + a_4, \qquad I_2 = \frac{-y}{2x^{\frac{1}{2}} y_x} + a_3 \ln(x) + a_4, \tag{17}$$
$$I_3 = \frac{-y\ln(y)}{2xy_x} + a_3 \ln(x) + a_4.$$

6 Classification of Lie algebra

The generating operators of the Lie symmetry group of (1) are presented in (3). The above indicates that the vector space generated by the operators described forms a 8-dimensional Lie algebra. In the next proposition we assume some particular criterion of semisimplicity given by Cartan, and its proof can be found in [17]

Proposition 6.1. (Cartan's theorem) A Lie algebra is semisimple if and only if its Killing form is nondegenerate.

Cartan's theorem is useful to classify the Lie algebras obtained in the symmetry analysis of differential equations, more specifically, for deciding whether these algebras are semisimple or not.

Proposition 6.2. The Lie algebra associated to (1) is a semisimple Lie algebra.

Proof. Let $\mathfrak{g}$ be the Lie algebra associated to (1). We use the Cartan criterion of semisimple Lie algebra. First we calculate the matrix corresponding to the Cartan-Killing form. The matrix of the Cartan-Killing form is given by

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -18 & 0 \\ 0 & 0 & 0 & 0 & 0 & -18 & 0 & 0 \\ 0 & 0 & 0 & 54 & 0 & 0 & 0 & 0 \\ 0 & 0 & 54 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 108 & 0 & 0 & 54 \\ 0 & -18 & 0 & 0 & 0 & 0 & 0 & 0 \\ -18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 54 & 0 & 0 & 28 \end{bmatrix}.$$

A straightforward computation shows that det $(M) \neq 0$ so, by the Cartan criterion, $\mathfrak{g}$ is a semisimple Lie algebra, where $\mathfrak{g}$ is the Lie algebra generated by the vector field of the symmetry group that is obtained from (1). After this we compute the signature and we get that it is given by $(5, 3)$. $\square$

One way to completely determine a Lie algebra is given by by its roots system, since it is known that a semisimple Lie algebra is uniquely determined by such system. In the next proposition we find a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Definition 6.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an arbitrary field k. Choose a basis e_j, $1 \leq i \leq n$, in $\mathfrak{g}$ where $n = \dim \mathfrak{g}$ and set $[e_i, e_j] = C_{ij}^k e_k$. Then the coefficients C_{ij}^k are called structure constants.

They form a structure tensor, which is an element of the space $\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}$.

Proposition 6.3. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras of dimension n. Suppose each has a basis with respect to which the structure constant are the same. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic.

Proposition 6.4. Let $\mathfrak{g}_3$ be the Lie subalgebra that has a basis given by the vector field Π_3, Π_4, Π_5. Then $\mathfrak{g}_3$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Proof. Let's define H, X, Y by $H = \dfrac{1}{3}\Pi_5$, $X = \dfrac{1}{3}\Pi_3$, $Y = \dfrac{1}{3}\Pi_4$, then $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. Considering a change of basis suggested by the preceding expressions, we obtain constant structure that coincide with those of $\mathfrak{sl}(2, \mathbb{R})$ and consequently, by Proposition 6.3, the statement is proved. □

Theorem 6.1. Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exist ideals $\mathfrak{g}_1, \cdots, \mathfrak{g}_\mathfrak{r}$ of $\mathfrak{g}$ which are simple (as Lie algebras), such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\mathfrak{r}$. Every simple ideal of $\mathfrak{g}$ coincide with one of $\mathfrak{g}_\mathfrak{i}$. Moreover, the Killing form of $\mathfrak{g}_\mathfrak{i}$ is the restriction of k to $\mathfrak{g}_\mathfrak{i} \times \mathfrak{g}_\mathfrak{i}$.

Proposition 6.5. Let $\mathfrak{g}$ be the Lie algebra associated to (1). Then $\mathfrak{g}$ is isomorphic to the special linear Lie algebra of order 3, that is, $\mathfrak{sl}(3, \mathbb{R})$.

Proof. This proof is based on the Theorem 6.1. Due to that theorem the Lie algebra $\mathfrak{g}$ is semisimple by the Cartan criterion, the Proposition 6.2, and the classification of simple Lie algebras. In fact, if we begin with the first term of the decomposition of the semisimple Lie Algebra $\mathfrak{g}$, namely $\mathfrak{g}_1$, we get a simple Lie subalgebra and therefore the dimension of $\mathfrak{g}$ must be greater than 2, so we have two cases: $\mathfrak{sl}(2)$ or $\mathfrak{so}(3)$, in that order. For instance, if $\mathfrak{g}_1 = \mathfrak{sl}(2)$, which has dimension three, we need to consider other algebras to attain the dimension 8, but it is not possible precisely because of the dimension (since there are no 5 dimensional simple Lie algebras). Continuing with this argument, if we add another $\mathfrak{sl}(2)$, then it only remains to add a 2-dimensional simple Lie algebra, so all dimensions can sum up to 8, but such algebra two dimensional does not exist. Therefore, in this case, the Lie algebra is indecomposable and the only option is that the Lie algebra is $\mathfrak{sl}(3)$. In the other hand, If we start with the the first term of the decomposition as $\mathfrak{so}(3)$, we can consider a similar argument and get to the same conclusions.
Another important fact to be considered is that there are no 8 dimensional Lie algebra in the class of $\mathfrak{so}(n)$. Consequently, the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(3)$, and the statement is proved.

Remark 6.1. To be more clear, we start with the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, for instance, and then we choose the first $\mathfrak{g}_1$ to be a simple algebra of the least possible dimension. Then, we know that there are two 3-dimensional Lie algebras, which are: $\mathfrak{sl}(3)$ and $\mathfrak{so}(3)$. To complete the dimension of $\mathfrak{g}$, we consider other simple Lie algebras among the previously stated, and fill in the decomposition of $\mathfrak{g}$, looking always at the dimension that remains to be filled in. We continue adding algebras until we reach the dimension of $\mathfrak{g}$. This procedure forces $\mathfrak{g}$ to be equal to $\mathfrak{g}_1 = \mathfrak{sl}(3)$.

□

7 Conclusion

Using the Lie symmetry group of (3), we calculated the optimal algebra, as it was presented in Proposition 3.1. Making use of some elements of the optimal algebra, it was possible to obtain some invariant solutions as it was shown in Table 4. Both the invariant solutions of numeral $1, 2$ and the family of implicit invariant solutions corresponding to the numerals $5-10$ from Table 4, do not appear in the literature known until today.

It has been shown the variational symmetries for (1) in (16) with its corresponding conservation laws (17). The Lie algebra associated to the equation (1) is a semisimple algebra as it is proved in Proposition 6.2 and the signature of the Cartan-Killing form is $(5, 3)$. The Proposition 6.4 showed that the Lie algebra associated to the equation (1) has, at least, one Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Lastly, Proposition 6.5 showed that the Lie algebra associated to the equation (1) is isomorphic to $\mathfrak{sl}(3)$ and therefore, the goal initially proposed was achieved.

For future works, numerical methods could be used to solve the implicit equations shown in numerals $5-10$ from the Table 4. An alternative line of work would be to use the Lie symmetry group to calculate the λ-symmetries of (1), and, thus, explore more conservation laws for (1). Equivalence group theory could be also considered to obtain preliminary classifications associated to a complete classification of (1).

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